Topics in the Calculus of Variations: Recent Advances and New Trends

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1 Overview and Outcome of the Meeting

The calculus of variations is at the same time a classical area of mathematical analysis with longstanding open problems and a very active subject of modern mathematics, that has important applications in a variety of different fields, such as materials science, mathematical physics, and treatment of digitalized images, just to name a few examples. In the last decades this subject has enjoyed a flourishing development worldwide, driven both by mathematical developments and emergent applications.

In this workshop a special emphasis was given to young researchers. Indeed, out of the 31 talks, 2 were given by graduate students (Hagerty and Gravina), 12 by postdoctoral fellows (Carroccia, Cristoferi, Davoli, Ginster, Gladbach, Maor, Morandotti, Murray, O'Brien, Piovano, Tobasco, Wojtowytsch) and 3 by junior faculty (Friedrich, Iurlano, Rüland). These were complemented by talks from worldwide experts in different areas of calculus of variations and partial differential equations including Gianni Dal Maso (Gamma-convergence, fracture mechanics), Georg Dolzmann (elasticity, plasticity, microstructure), Nicola Fusco (regularity of partial differential equations, geometric measure theory), and Giuseppe Savaré (abstract evolution equations, optimal transport, analysis in metric-measure spaces). The presence of so many young participants and the wonderful environment of the Banff International Research Center contributed to a very informal, friendly, and unique atmosphere. We kept the talks to 35 minutes. This gave plenty of time for informal discussions and networking. New friendships were formed and collaborations were initiated during the meeting. There were several social activities, including an official hike along the Hoodoos Trail, as well as a 7 hours hike to Mount Rundle by some of the more adventurous participants. From the feedback we received, the workshop was very successful and many participants expressed the desire to have another one in the next few years.

Participants and especially young people were exposed to a wide range of topics at the cutting edge in nonlinear and applied analysis. Below we highlight some of them.

2 Epitaxial Growth

Four of the talks (Cristoferi, Lu, Piovano, and partially Fusco) were on epitaxial growth, which is the deposition of a crystalline film onto a substrate, in which the atoms of the film occupy natural lattice positions of the substrate. If the film and the substrate are of the same material, the epitaxial deposition is called *homoepitaxy*, while if they are of different material, it is called *eteroepitaxy*. In eteroepitaxial growth if the mismatch in lattice parameter of the film crystal and the substrate crystal is large, then the film tends to gather into islands on the surface and the substrate is exposed between islands. This is called the *Volmer-Weber (VW) growth mode*. On the other hand, if the mismatch in lattice parameter is small the atoms of the film tend to align themselves with those of the substrate, continuing its atomic structure. This is due to the fact that the energy gain associated with the chemical bonding effect is greater than the strain in the film. This layer-by-layer film growth mode is called the *Frank–van der Merwe (FM) growth mode*. However, as the film continues to grow, the stored strain energy per unit area of interface increases linearly with the film thickness. Eventually, the presence of such a strain renders a flat layer of the film morphologically unstable or metastable, after a critical value of the thickness is reached. To release some of the elastic energy due to the strain, the atoms on the free surface of the film tend to rearrange into a more favorable configuration. Typically, after entering the instability regime, the film surface becomes wavy or the material agglomerates into clusters or isolated islands on the substrate surface. Island formation in systems such as In-GaAs/GaAs or SiGe/Si turns out to be useful in the fabrication of modern semiconductor electronic and optoelectronic devices such as quantum dots laser.

This growth mode in which islands are separated by a *thin wetting layer* is known as the *Stranski-Krastanow (SK) growth mode*.

In the literature several atomistic and continuum theories for the growth of epitaxially strained solid films are available. In [30] Spencer proposed the following variational approach in which the film and the substrate are modeled as linearly elastic solids. Consider an epitaxial layer (with variable thickness h) grown on a flat semi-infinite substrate. Restricting attention to two-dimensional morphologies which correspond to three-dimensional configurations with planar symmetry, assume that the material occupies the infinite strip

$$\Omega_h := \{\boldsymbol{x} = (x, y) : 0 < x < b,\, y < h(x)\} \tag{1}$$

where $h : [0, b] \to [0, \infty)$ is a Lipschitz function. Thus the graph of h represents the *free* profile of the film, the open set and the line $y = 0$ corresponds to the film/substrate interface. We work within the theory of small deformations, so that $\boldsymbol{E}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right)$ represents the strain, with $\boldsymbol{u} : \Omega_h \to \mathbb{R}^2$ the planar displacement. The displacement is measured from a configuration of the layer in which the lattices of the film and the layer are perfectly matched; this configuration, in which $\boldsymbol{E} \equiv \boldsymbol{0}$, will not correspond to a minimum energy state of the film, which we assume to occur at a strain $\boldsymbol{E}_0 = \boldsymbol{E}_0(y)$. If the film and the substrate have similar material properties, then they share the same homogeneous elasticity tensor C. Hence, bearing in mind the mismatch, the elastic energy per unit area is given by $W(\boldsymbol{E} - \boldsymbol{E}_0(y))$, where

$$W(\boldsymbol{E}) := \frac{1}{2}\boldsymbol{E} \cdot C[\boldsymbol{E}] \tag{2}$$

with C a positive definite fourth-order tensor.

In the sharp interface model the interfacial energy density ψ has a step discontinuity at $y = 0$: It is $\gamma_{\text{film}} > 0$ if the film has positive thickness and $\gamma_{\text{sub}} > 0$ if the substrate is exposed, to be precise,

$$\psi(y) := \begin{cases} \gamma_{\text{film}} & \text{if } y > 0, \\ \gamma_{\text{sub}} & \text{if } y = 0. \end{cases} \tag{3}$$

Hence the total energy of the system is given by

$$\mathcal{F}(\boldsymbol{u}, h) := \int_{\Omega_h} W(\boldsymbol{E}(\boldsymbol{u}) - \boldsymbol{E}_0)\, d\boldsymbol{x} + \int_{\Gamma_h} \psi\, ds, \tag{4}$$

where Γ_h represents the free surface of the film, that is,

$$\Gamma_h := \partial\Omega_h \cap ((0, b) \times \mathbb{R}). \tag{5}$$

Most of the recent literature (see [24] and the references therein) deals with the Stranski-Krastanow (SK) growth mode, which corresponds to the case $\gamma_{\text{film}} < \gamma_{\text{sub}}$. In his talk, Paolo Piovano (University of Vienna) presented some recent work [8] in collaboration with Elisa Davoli (University of Vienna), another of the workshop speakers, in the Volmer-Weber (VW) growth mode $\gamma_{\text{film}} > \gamma_{\text{sub}}$ and in the case in which the film and the substrate have different homogeneous elasticity tensor C. The Young-Dupré law is shown to be satisfied by the angle that energetically-optimal profiles form at contact points with the substrate. This is one of the first analytical validations of such relation, which was originally formulated in Fluid Mechanics, in the context of Continuum Mechanics for a thin-film model.

Riccardo Cristoferi (Carnegie Mellon University) and Xin Yang Lu (Lakehead University) in their talks considered modified variational models that take into consideration the effect of the free atoms moving on the surface (adatoms).

3 Ferromagnetism

Ferromagnetic materials have the physical property of exhibiting spatially ordered magnetization patterns (*magnetic domains*) under a variety of conditions. This property is at the basis of the large use of ferromagnets in technological applications. The mechanisms behind the magnetic domain formation can be quite complex, but usually domain patterns may be understood from energetic considerations based on the micromagnetic modeling framework. Ground states of various ferromagnetic systems have been widely studied in the physics community and more recently in the mathematical literature (see the review [11]). In particular, within the micromagnetic framework the ground state domain structure of macroscopically thick uniaxial ferromagnetic films is by now fairly well understood mathematically. In contrast, most mathematical treatments of microscopically thin ferromagnetic films deal with the case where the magnetization prefers to lie in the film plane. Thus, one of the main open questions in the theory of uniaxial ferromagnets is to rigorously characterize their ground states in the case of films of vanishing thickness where the magnetization prefers to align normally to the film plane. This was the subject of Hans Knüpfer's talk, who reported on a joint work [21] with C. Muratov (another of the workshop speakers) and F. Nolte. Experimental observations show the formation of bubble and stripe domain patterns in thin ferromagnetic films with strong perpendicular anisotropy. Using rigorous analysis, Knüpfer and collaborators identified the critical scaling at which the phase transition from a single domain state to multi-domain states occurs. Moreover, they derived a two-dimensional effective model in the single domain regime and a scaling law for the minimal energy in the multidomain regime.

4 Free Boundary and Obstacle Problems

Cagnetti, Gravina, and Rüland talked about free boundary and thin obstacle problems. Many problems in imaging, materials science, physics, and other areas can be described by partial differential equations that exhibit apriori unknown sets, called free boundaries, such as interfaces and moving boundaries. The study of such sets occupies a central position in such problems. A classical example is the one-phase free boundary problem studied by Alt and Caffarelli in [2], to be precise,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap \{u > 0\}, \\ u = 0 & \text{on } \Omega \cap \partial\{u > 0\}, \\ |\nabla u| = Q & \text{on } \Omega \cap \partial\{u > 0\}, \\ u = u_0 & \text{on } \Gamma. \end{cases} \tag{6}$$

Here Ω is an open connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary and Q is a nonnegative measurable function. Solutions to (6) are critical points for the functional

$$J(u) := \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}} Q^2\right) dx, \quad u \in \mathcal{K}, \tag{7}$$

where $\mathcal{K} := \{u \in H^1_{\text{loc}}(\Omega) : u = u_0 \text{ on } \Gamma\}$, with $\Gamma \subset \partial\Omega$ a measurable set with $\mathcal{H}^{N-1}(\Gamma) > 0$ and $u_0 \in H^1_{\text{loc}}(\Omega)$ a nonnegative function satisfying $J(u_0) < \infty$. The equality $u = u_0$ on Γ is in the sense of traces. Under the assumption that Q is a Hölder continuous function satisfying

$$0 < Q_{\min} \leq Q(x) \leq Q_{\max} < \infty, \tag{8}$$

Alt and Caffarelli proved local Lipschitz regularity of local minima and showed that the free boundary $\partial\{u > 0\}$ is a $C^{1,\alpha}_{\text{loc}}$ regular curve in Ω if $N = 2$, while if $N \geq 3$ they proved that the reduced free boundary is a hypersurface of class $C^{1,\alpha}_{\mathrm{loc}}$ in Ω, for some $0 < \alpha < 1$. The regularity of the free boundary is strongly related to the assumption $0 < Q_{\min} \leq Q(x)$. Indeed, in the special case in which $\Omega = R \times (0, \infty)$, with $R \subset \mathbb{R}^{N-1}$ a cube, $Q(x) = (h - x_N)_+$, $u_0(x) = (m - x_N)_+$, and $Q \times \{0\} \subset \Gamma$, the free boundary problem (6) is related to water waves and to Stokes' conjecture of water waves of greatest height (see [29] and the references therein). In his talk, Ph.D. student Giovanni Gravina (Carnegie Mellon University) showed that by taking $\Gamma = (Q \times \{0\}) \cup (\partial Q \times (\gamma, \infty))$ minimizers of (2) are not one-dimensional for a suitable choice of the parameters m and γ. This is a first crucial step towards a variational proof of Stokes' conjecture (see [20], see also [13]).

The speaker Angkana Rüland (Max Planck Institute for Mathematics, Leipzig) discussed higher regularity of the free boundary for the thin obstacle problem with variable coefficients, to be precise

$$\sum_{i,j=1}^{N} \int_{B^+} a_{ij} \partial_i u \partial_j u \, dx$$

in $\mathcal{K} := \{u \in H^1(B^+) : u \geq 0 \text{ on } B' \times \{0\}\}$, where $B^+ := B(0,1) \cap \{x_N > 0\}$ and $B' := \{x' \in \mathbb{R}^{N-1} : |x'| < 1\}$, and the coefficients a_{ij} are symmetric, satisfy standard uniform ellipticity conditions, and are sufficiently regular. Together with her collaborators in [22] she proved that if the coefficients a_{ij} belong to $W^{1,p}$ for $p > N$, then the regular free boundary is of class $C^{1,1-N/p}$, while if the coefficients a_{ij} belong to $C^{k,\alpha}$ for $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, then the regular free boundary is of class $C^{k,\alpha}$, and, finally, that if the coefficients a_{ij} are analytic, then so is the regular free boundary.

5 Gradient Flows and Abstract Evolution Equations

The speakers Dal Maso, Fusco, Lu, Morini, Savaré discussed about gradient flows and abstract evolution equations with applications to crystalline mean curvature flows, epitaxial growth, and wave equations in time-dependent domains.

The variational notion of *minimizing movements* was introduced by De Giorgi in [9] to present a general and unifying approach to a large class of evolution problems. Given a topological space X, a functional $\Phi : (0, \infty) \times X \times X \to \mathbb{R}$, an initial datum $u_0 \in X$, and a discrete time step $\tau > 0$, one looks for sequences $\{u^n_\tau\}_n$ such that $u^0_\tau = u_0$ and u^n_τ is defined recursively as the minimizer of the functional

$$x \in X \mapsto \Phi(\tau, x, u^{n-1}_\tau),$$

if it exists. Setting

$$u_\tau(t) := \begin{cases} u^0_\tau & \text{if } t = 0, \\ u^{n-1}_\tau & \text{if } (n-1)\tau < t \leq n\tau, \, n \in \mathbb{N}, \end{cases}$$

a function $u : (0, \infty) \to X$ is called *generalized minimizing movement* associated to Φ and u_0 if there exist a subsequence $\{\tau_k\}_k$ such that $\tau_k \to 0^+$ and corresponding functions $u_{\tau_k} : (0, \infty) \to X$ such that $u_{\tau_k} \to u$ pointwise in $(0, \infty)$.

In [9] De Giorgi conjectured that if $X = \mathbb{R}^N$, $\phi : \mathbb{R}^N \to \mathbb{R}$ is a continuously differentiable and Lipschitz function, and $u_0 \in \mathbb{R}^N$, a function $u \in C^1([0, \infty); \mathbb{R}^N)$ is a solution of the Cauchy problem for the gradient flow

$$\begin{cases} u'(t) = -\nabla \phi(u(t)), \\ u(0) = u_0, \end{cases}$$

if and only if there exist a family of Lipschitz continuous functions $\phi_\tau : \mathbb{R}^N \to \mathbb{R}$, $\tau > 0$, such that $\mathrm{Lip}(\phi_\tau - \phi) \to 0$ as $\tau \to 0^+$ and u is the generalized minimizing movement associated to the function

$$\Phi(\tau, x, y) := \frac{1}{2\tau}|x - y|^2 + \phi_\tau(x), \quad \tau > 0, \quad x, y \in \mathbb{R}^N.$$

In his blackboard talk Giuseppe Savaré (University of Pavia) described how he solved this conjecture in [12].

The speaker Gianni Dal Maso (SISSA, Italy) discussed second order abstract evolution equations in moving domains of the form

$$\begin{cases} u''(t) + Au(t) = 0 & \text{for a.e. } t > 0, \\ u(t) \in V_t & \text{for a.e. } t > 0, \\ u(0) = u_0, \, u'(0) = u_1. \end{cases}$$

Here $\{V_t\}_t$ is an increasing sequence of spaces contained in a common Hilbert space H, $u_0 \in V_0$, $u_1 \in H$, and A is a linear continuous and coercive operator mapping V_t into its dual V_t'. In applications to fracture mechanics, $H = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^N$ and $V_t = H^1(\Omega \setminus \Gamma_t)$, where Γ_t is an $(N-1)$-dimensional closed set representing the crack at time t. Existence of weak solutions is obtained in [7] using an approach suggested by De Giorgi and developed by Serra and Tilli for the wave equation. Weak solutions are obtained as limits of minimizers of the family of functionals

$$\mathcal{F}_\varepsilon(u) = \frac{1}{2}\int_0^\infty e^{-t/\varepsilon}(\varepsilon^2\|u''(t)\|_H^2 + a(u(t), u(t)))\,dt$$

defined on an appropriate space.

The speaker Nicola Fusco (University of Naples) discussed a surface diffusion problem of the type

$$V_t = \kappa\Delta_{\Gamma_t}(\operatorname{div}_{\Gamma_t}\nabla\varphi(\nu_t) - W(E(u_t))),$$

which describes the evolution of voids in a crystalline material. Here φ is the anisotropic surface density, Γ_t is the evolving surface, $E(u_t)$ is the trace of the infinitesimal strain at time t, and Δ_{Γ_t} and $\operatorname{div}_{\Gamma_t}$ are the tangential Laplacian and divergence. This evolution equation is the gradient flow with respect to H^{-1} of the energy

$$F \mapsto \int_{\Omega\setminus F} W(E(u))\,dx + \int_{\partial F}\varphi(\nu_F)\,ds,$$

where F is the void. In the recent paper [18] it was proved that if the initial configuration is stable then the solution exists for all time. This is the first existence result for this type of geometric motion without the presence of a higher order regularizing term involving the curvature in the energy.

The speaker Morini (University of Parma) considered the gradient flow of the same anisotropic surface energy $F \mapsto \int_{\partial F}\varphi(\nu_F)\,ds$ with respect to L^2. This gives rise to the motion by mean curvature

$$V_t = -m(\nu_t)\kappa_{F_t},$$

where $m(\nu_t)$ is the mobility and k_{F_t} is the mean curvature, which can be written as $\kappa_{F_t} = \operatorname{div}_{\Gamma_t}\nabla\varphi(\nu_t)$, when F_t is smooth. The well-posedness and the validity of the maximum principle for this type of evolution equations had been a long standing open problem, which was finally solved in [5].

6 Phase Transitions Problems

The speakers Davoli, Hagerty, and Murray addressed various problems in liquid-liquid and solid-solid phase transitions. The prototype for these kind of singularly perturbed problems is the van der Walls–Cahn–Hilliard theory of phase transitions. Consider a fluid confined into a container $\Omega \subset \mathbb{R}^N$. Assume that the total mass of the fluid is m, so that admissible density distributions $u : \Omega \to \mathbb{R}$ satisfy the constraint $\int_\Omega u(x)\,dx = m$. The total energy is given by the functional $u \mapsto \int_\Omega W(u(x))\,dx$, where $W : \mathbb{R} \to [0, \infty)$ is the energy per unit volume. Assume that W supports two phases $a < b$, that is, W is a *double-well potential*, with

$$\{z \in \mathbb{R} :\, W(z) = 0\} = \{a, b\}. \tag{9}$$

Then any density distribution u that renders the body stable in the sense of Gibbs is a minimizer of the following problem

$$\min\left\{\int_\Omega W(u(x))\,dx :\, \int_\Omega u(x)\,dx = m\right\}. \tag{10}$$

If $a|\Omega| < m < b|\Omega|$, then given any measurable set $E \subset \Omega$ with

$$|E| = \frac{m - a|\Omega|}{b - a}, \tag{11}$$

the function $u = b\chi_E + a\chi_{\Omega\setminus E}$ is a solution of problem (10). This lack of uniqueness is due the fact that interfaces between the two phases a and b are not penalized by the total energy. The physically preferred solutions should be the ones that arise as limiting cases of a theory that penalizes interfacial energy, so it is expected that these solutions should minimize the surface area of $\partial E \cap \Omega$.

In the van der Walls–Cahn–Hilliard theory of phase transitions, the energy depends not only on the density u but also on its gradient, precisely,

$$G_\varepsilon(u) := \int_\Omega (W(u) + \varepsilon^2|\nabla u|^2)\, dx, \quad u \in H^1(\Omega). \tag{12}$$

Note that the gradient term penalizes rapid changes of the density u, and thus it plays the role of an interfacial energy. Stable density distributions u are now solutions of the minimization problem

$$\min\left\{\int_\Omega (W(u) + \varepsilon^2|\nabla u|^2)\, dx\right\}, \tag{13}$$

where the minimum is taken over all smooth functions u satisfying $\int_\Omega u(x)\, dx = m$. In 1985 Gurtin conjectured that the limits, as $\varepsilon \to 0$, of solutions u_ε of (13) are solutions u_0 of (10) with minimal surface area, that is, if $u_0 = a\chi_{E_0} + b\chi_{\Omega\setminus E_0}$, then

$$\text{surface area of } E_0 \leq \text{surface area of } E \tag{14}$$

for every measurable set with $|E| = \frac{m-a|\Omega|}{b-a}$. Moreover, he also conjectured that

$$G_\varepsilon(u_\varepsilon) \sim \varepsilon \text{ surface area of } E_0. \tag{15}$$

Using results of Modica and Mortola this conjecture was proved independently for $N \geq 2$ by Modica [27] and by Sternberg [31] using Gamma-convergence.

In his talk Ryan Murray (Penn State) presented the solution of a long standing open problem, the asymptotic development of order 2 by Gamma-convergence of the mass-constrained Cahn–Hilliard functional (12) (see [25]) and its applications to the slow motion of interfaces for the mass preserving Allen–Cahn equation and the Cahn–Hilliard equations in higher dimension.

The corresponding problem for gradient vector fields, where in place of G_ε we introduce

$$I_\varepsilon(\boldsymbol{u}) := \int_\Omega \left(W(\nabla\boldsymbol{u}) + \varepsilon^4|\nabla^2\boldsymbol{u}|^2\right) d\boldsymbol{x}, \quad \boldsymbol{u} \in W^{2,2}\left(\Omega;\mathbb{R}^d\right),$$

arises naturally in the study of elastic solid-to-solid phase transitions. Here $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ stands for the deformation. One of the main differences with the functional G_ε is that in the case of gradients, some geometrical compatibility conditions must exist between the wells. The speaker Elisa Davoli (University of Vienna), in collaboration with the speaker Manuel Friedrich (University of Münster) considered in dimension $N = 2$ the modified functional

$$J_\varepsilon(\boldsymbol{u}) := \int_\Omega \left(W(\nabla\boldsymbol{u}) + \varepsilon^4|\nabla^2\boldsymbol{u}|^2 + \varepsilon^2\eta_\varepsilon^2(|\partial^2_{11}\boldsymbol{u}|^2 + |\partial^2_{12}\boldsymbol{u}|^2)\right) d\boldsymbol{x}, \quad \boldsymbol{u} \in W^{2,2}\left(\Omega;\mathbb{R}^d\right),$$

where the new term $|\partial^2_{11}\boldsymbol{u}|^2 + |\partial^2_{12}\boldsymbol{u}|^2$ represents a penalization in the direction e_2 and $\eta_\varepsilon \to \infty$ as $\varepsilon \to 0^+$. The main result obtained were a two-well rigidity estimate, compactness of equibounded sequence, and the characterization of the effective limiting model, where the effective linearized energy involves elastic energy as well as two surface terms, one for the jumps of ∇u , which represent the energy associated to single phase transitions between the wells A and B. The second surface term corresponds to two consecutive phase transitions with a small intermediate layer. It enters the energy functional with double cost with respect to single phase transitions.

7 Plasticity and Fracture

One of the major subjects of the workshop was the study of failure phenomena in solids, such as plasticity, crack propagation, damage, and their microscopic mechanisms.

Plasticity is the property of a material that can undergo permanent deformations in response to an applied force. In the last decade the mathematical treatment of plasticity at the continuum scale has received a renewed interest, motivated by a change of perspective. While in the seminal work [32] quasistatic evolution was seen as the limit of viscoplastic evolutions when the viscosity parameter tends to zero, in [6] the classical theory of Prandtl-Reuss plasticity was revisited within the modern framework of rate-independent systems [26]. The advantage of this formulation is that quasistatic evolution is now viewed as a time-parameterized family of minimization problems, so that very mild regularity is needed in the definition of solutions and variational methods can be used to prove existence.

Dynamic evolution in plasticity was discussed in the talk by Jean-François Babadjian (Paris Orsay). On the one hand, using variational methods, one can show that the problem of dynamic perfect plasticity is well-posed in a suitable measure theoretic setting. On the other hand, the problem can be formulated as a constrained boundary value Friedrichs' system. He showed that the variational solution coincides with the unique entropic solution of the hyperbolic formulation and, owing to the finite speed propagation property, he established a new short time regularity result for the solution.

Plasticity in metals is considered to be the macroscopic consequence of the presence (and motion) of curve-like defects in the atomic structure, called dislocations: plastic deformation is the overall result of the relative slip of atomic layers, which is favored by the presence of dislocations. Any predictive theory of plasticity at the continuum scale should therefore take into account the presence and the motion of these defects. Although several models are available in the engineering literature to describe the dislocation behavior at the microscopic scale, it is not yet clear how to include the effect of their presence and motion in a model at the macroscopic level. In the last years this issue has been the object of intense research in the engineering and in the mathematical community. In particular, the approach by Gamma-convergence has led to rigorous upscaling results, both in the static and in the evolutionary setting, see, e.g., [1, 19, 28]. However, these contributions, as well as the majority of the mathematical literature, only apply to an idealized setting, where dislocations are modeled as straight and parallel lines. Although the mathematical challenges are still numerous, this modeling assumption strongly reduces the complexity of the problem: in reality dislocations are curves in three dimensions, that typically form loops, entangle with one another, and get pinned at obstacles. Janusz Ginster (Carnegie Mellon University) presented a very recent work, where the equilibrium problem for a curved dislocation line in a three-dimensional domain is considered. Using a core radius regularization and sending the core radius to zero, he derived an asymptotic expression for the induced elastic energy. He then deduced the expression of the force acting on the dislocation line by computing the variation of the induced elastic energy. In his lecture Georg Dolzmann (University of Regensburg) discussed how to describe crystals with one active slip system in a continuum plasticity model.

Fracture processes have been widely studied in the last twenty years, starting from the pioneering work of Griffith and the variational approach of Francfort-Marigo [14], where crack evolution is determined through a competition between the surface energy spent to increase the crack and the corresponding release of the bulk elastic energy. Damage corresponds to the worsening of the elastic properties of a material as a consequence of applied stresses. There is an obvious connection between damage and fracture: on the one hand, a large number of microcracks may weaken the elastic properties of the material; on the other hand, concentration of damage may macroscopically results in a fracture. The classical phase-field approximation result of the Mumford-Shah functional by Ambrosio-Tortorelli [3] can be reinterpreted in this spirit, where the phase-field variable describes the local amount of damage. An overview on this topic was given in the talk by Marco Caroccia (Lisbon), who also reported on a new approximation result for a fluid-driven fracture model. The speaker Flaviana Iurlano (University of Paris 6) showed existence of strong minimizers for the Griffith model in the formulation by Francfort and Marigo, in the framework of linearized elasticity. A strong minimizer is a minimizer given by a function defined in an open set with a closed discontinuity set of codimension one. What makes this variational problem hard to tackle is that its natural formulation takes place in the space SBD of special functions with bounded deformation, where standard approximation methods based on truncation and on the coarea formula do not apply. In the recent paper [4] it was shown that SBD functions with small jump sets are close in energy to functions which are smooth in a slightly smaller domain. This result is a key ingredient in proving existence of strong minimizers for the Griffith's problem and may prove useful in several applications, including the study of quasistatic crack evolution.

8 Thin Structures and Pattern Formation

Since the beginning of research in elasticity a large body of work has been devoted to the justification of lower dimensional theories for thin structures in terms of the fully three-dimensional theory. A *thin structure* is a three-dimensional body, whose thickness in one or two directions is very small compared with the remaining dimensions, such as a rod, a membrane, a plate, or a shell. These kinds of structures are ubiquitous in the physical world. A precise understanding of the laws governing their equilibrium configurations is therefore crucial in a large number of applications, ranging from aerospace and civil engineering to biology.

In the classical approach lower dimensional theories are usually obtained by formal asymptotic expansions, and hence their range of validity is typically unclear. In the early 90's a rigorous approach based on Gamma-convergence has emerged and, starting from the seminal papers [16, 17], has led to the identification of a hierarchy of limit models for plates, rods, and shells. In his talk Marco Morandotti (Technical University Munich) discussed how to perform dimension reduction within the framework of structured deformations. Structured deformations [10] provide a rich geometrical setting that includes not only the smooth, classical deformations that underlie much of solid mechanics, but also the piecewise smooth deformations that describe macroscopic cracking in fracture mechanics, as well as the complex combination of macroscopic and microscopic changes relevant for the study of crystals with defects and granular materials.

Recently, there has been a growing interest in the study of prestrained elastic plates, that is, plates that do not have a stress-free configuration. These plates, also known as non-Euclidean plates, can be generated via growth, plastic deformation, or active swelling, which are nonuniform across the sheet. Within the formalism of incompatible elasticity such nonuniform deformations prescribe a non-Euclidean reference metric field on the plate. Because of this geometric incompatibility non-Euclidean plates assume non-trivial equilibrium configurations in the absence of exterior forces or imposed boundary conditions. This is a desired property in applications, for instance in the design of self-shaping bodies. Mathematically, minimizing the elastic energy for a non-Euclidean plate corresponds to the question of finding the "most isometric" immersion of a Riemannian manifold into another one of the same dimension. The speaker Cy Maor (University of Toronto) showed how this question relates to a generalization of Reshetnyak's rigidity theorem to Riemannian manifolds.

Another topic of the workshop was the study of energy-driven pattern formation in thin sheets. In the last years the wrinkling of thin elastic sheets has attracted a lot of attention in both the mathematics and the physics communities. A general feature of these problems is that wrinkling arises as an energetically preferable alternative to compression. In particular, a growing literature is developing on the scaling law of the elastic energy, i.e., its dependence on the sheet thickness h, for problems involving wrinkling. Wrinkled configurations can be viewed as (local) minimizers of a suitable elastic energy E^h, consisting of a non-convex membrane energy plus a higher order singular perturbation representing bending energy. It is intuitively clear that the bending energy is small in h: stretching a thin sheet ought to take much more energy than bending it. On the other hand, the membrane energy alone is non-convex, so, if the bending energy is ignored, then there might not exist a minimizer. Computing the relaxed energy gives partial information about the minimizers of E^h, but any feature that vanishes in the limit $h \to 0$ is invisible to the relaxed problem. This is where energy scaling laws may prove useful to have more quantitative bounds on the minimal energy and on fine properties of the wrinkles. In the talk by Ethan O'Brien (Carnegie Mellon University) a specific system in the mechanics of thin elastic sheets was explored [23], in which geometry and loading conspire to generate fine-scale wrinkling, and the optimal energy scaling was determined.

Pattern formation in graphene was the subject of the talk by Manuel Friedrich (University of Münster). Graphene is a one-atom thick layer of carbon atoms arranged in a regular hexagonal lattice. Despite the progressive growth of experimental, computational, and theoretical understanding of graphene, the accurate description of its fine geometry is still elusive. Observations on suspended samples seem to indicate that graphene is generally not exactly flat but gently rippled. On the other hand, free graphene samples in absence of support have the tendency to roll-up in tube-like structures. Friedrich reported on a recent work [15], where a complete classification of ground-state deformations of the hexagonal lattice with respect to configurational energies including two- and three-body terms was provided. He showed that all energy minimizers are either periodic in one direction, as in the case of ripples, or rolled up, as in the case of nanotubes. For suspended samples the analysis can be further refined and the emergence of wave patterning can be proved.

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