Stability of the Gaussian Isoperimetric Problem

Marco Barchiesi

Università di Napoli Federico II

BIRS – May 2018

- Symmetry of minimizers of a Gaussian isoperimetric problem

  with Vesa Julin
- Sharp dimension free quantitative estimates for the gaussian isoperimetric inequality, *Ann. Probab.* (2017)

  with Vesa Julin and Alessio Brancolini

http://cvgmt.sns.it/people/barchiesi/

Gaussian isoperimetric inequality

Gauss space is $\mathbb{R}^n$ with measure

$$\gamma(E) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_E e^{-\frac{|x|^2}{2}} dx$$

for every $E \subset \mathbb{R}^n$. It is a probability measure, $\gamma(\mathbb{R}^n) = 1$.

Gaussian surface measure or Gaussian perimeter

$$P_\gamma(E) := \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\partial E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x)$$

when E sufficiently regular. We will use the notation

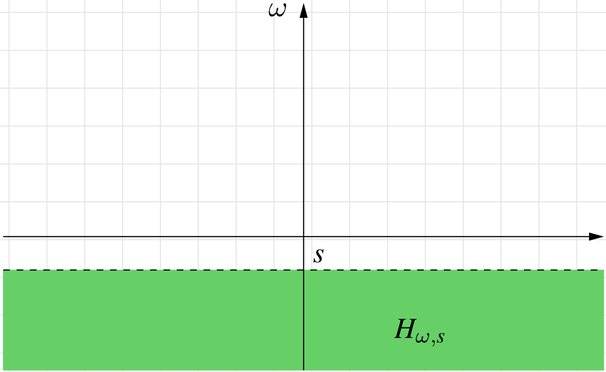
$$\mathcal{H}_\gamma^{n-1} = e^{-\frac{|x|^2}{2}} \mathcal{H}^{n-1}.$$

Gaussian isoperimetric inequality

Among all sets with given Gaussian measure, the half-space has the smallest Gaussian perimeter.

Some notation

- $H_{\omega,s} := \{x \in \mathbb{R}^n : x \cdot \omega < s\}, \quad \omega \in \mathbb{S}^{n-1}$
- $\phi(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}} dt.$



Some other notation

- $H_{\omega,s} := \{x \in \mathbb{R}^n : x \cdot \omega < s\}, \quad \omega \in \mathbb{S}^{n-1}$
- $\phi(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}} dt.$

Observe that in every dimension

$$\gamma(H_{\omega,s}) = \phi(s) \qquad \text{and} \qquad P_\gamma(H_{\omega,s}) = e^{-s^2/2}.$$

Theorem (Gaussian isoperimetric inequality)

For every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ it holds

$$P_\gamma(E) \geq e^{-s^2/2},$$

and the equality holds if and only if $E = H_{\omega,s}$ for some $\omega \in \mathbb{S}^{n-1}$.

A lot of proofs... Sudakov-Tsirelson (1974), Borell (1975), Carlen-Kerce (2001). The latter characterizes the extemals.

Symmetric case

The half-space is not symmetric. So a natural question is this.

Question: "Among all sets with given Gaussian measure, what is the symmetric set with the smallest Gaussian perimeter?"

Easy question, hard answer: at the moment we have no a precise idea about the possible shape of the solution. One of the main difficulties is that symmetrization techniques fail (I mean, we failed in using them).

We go along a different path...

Stability

Question: How much is positive the following quantity?

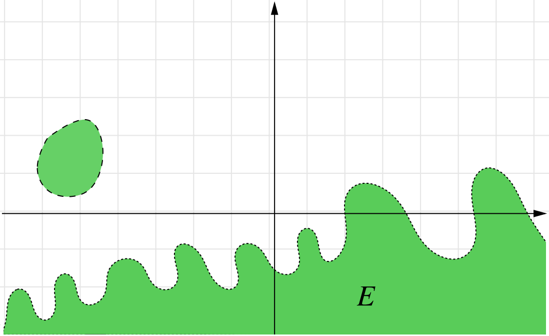
$$P_\gamma(E) - e^{-s^2/2}$$

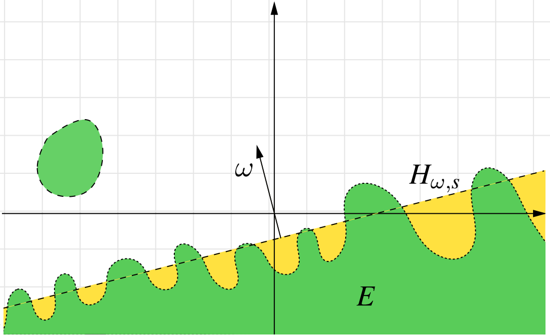
Theorem (Cianchi-Fusco-Maggi-Pratelli (2011))

For every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ it holds

$$P_\gamma(E) - e^{-s^2/2} \geq c_{n,s}\, \alpha(E)^2$$

for some constant $c_{n,s}$ depending both on the dimension n and the volume $\phi(s)$. Here $\alpha(E) := \min_{|\omega|=1} \gamma(E \Delta H_{\omega,s})$


E



ω
$H_{\omega,s}$
E

Theorem (Cianchi-Fusco-Maggi-Pratelli (2011))

For every set $E \subset \mathbb{R}^n$ *with* $\gamma(E) = \phi(s)$ *it holds*

$$P_\gamma(E) - e^{-s^2/2} \geq c_{n,s}\, \alpha(E)^2$$

for some constant $c_{n,s}$ *depending both on the dimension* n *and the volume* $\phi(s)$*. Here* $\alpha(E) := \min_{|\omega|=1} \gamma(E \Delta H_{\omega,s})$

- \+ The decay rate is sharp.
- \- The constant should not depend on the dimension (here $\sim 2^n$).

Theorem (Mossel-Neeman (2013))

For every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ holds

$$P_\gamma(E) - e^{-s^2/2} \geq c_s\, \alpha(E)^{4+\varepsilon}$$

for some constant c_s depending only on the volume $\phi(s)$.

- The decay rate is not sharp.
+ The constant does not depend on the dimension.

Theorem (Eldan (2015))

For every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ holds

$$P_\gamma(E) - e^{-s^2/2} \geq c_s\, \beta(E) \big| \log \beta(E) \big|^{-1}$$

for some constant c_s depending only on the volume $\phi(s)$. Here $\beta(E) := \min_{|\omega|=1} \big| b(E) - b(H_{\omega,s}) \big|$ and $b(E) := \int_E x\, d\gamma$.

The asymmetry β is stronger since it controls the standard α as

$$\beta(E) \geq \frac{e^{\frac{s^2}{2}}}{4}\, \alpha(E)^2.$$

Theorem (B-Brancolini-Julin (2017))

For every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ holds

$$P_\gamma(E) - e^{-s^2/2} \geq \frac{c}{1+s^2}\,\beta(E)$$

for some absolute constant c.

+ The decay rate is sharp.
+ The constant does not depend on the dimension.
+ The dependence on the volume is optimal.
- The constant c is not sharp.

Theorem (B-Brancolini-Julin (2017))

For every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ holds

$$P_\gamma(E) - e^{-s^2/2} \geq \frac{c}{1+s^2}\,\beta(E)$$

for some absolute constant c.

+ The decay rate is sharp.
+ The constant does not depend on the dimension.
+ The dependence on the volume is optimal.
- The constant c is not sharp.

The barycenter

The half-space maximizes the lenght of the barycenter: if $\gamma(E) = \phi(s)$, then

$$b_s := |b(H_{\omega,s})| \geq |b(E)|.$$

Moreover the asymmetry $\beta(E)$ is not obtained via a minimum problem.

$$\beta(E) := b_s - |b(E)|.$$

A new approach

We consider the functional

$$\mathcal{F}(E) = P_\gamma(E) + \varepsilon\sqrt{\pi/2}|b(E)|^2, \qquad \gamma(E) = \phi(s)$$

Remark

In minimizing $\mathcal{F}$ the two terms $P_\gamma(E)$ and $|b(E)|$ are in competition. Minimizing $P_\gamma(E)$ means to push the set E at infinity in one direction, so that it becomes closer to a half-space. On the other hand, minimizing $|b(E)|$ means to balance the volume of E with respect to the origin. For ε small enough the perimeter term overcomes the barycenter, and the only minimizers of $\mathcal{F}$ are the half-spaces $H_{\omega,s}$.

Old result

Theorem (B-Brancolini-Julin (2017))

The only minimizers of the functional $\mathcal{F}$ are the half-spaces when when $\varepsilon > 0$ is small.

Question: "What does it happen when ε is not longer small? Does the barycenter term win?"

Old result

Theorem (B-Brancolini-Julin (2017))

The only minimizers of the functional $\mathcal{F}$ are the half-spaces when when $\varepsilon > 0$ is small.

Question: “What does it happen when ε is not longer small? Does the barycenter term win?”

Some other notation

- $D_{\omega,s} := \{x \in \mathbb{R}^n : |x \cdot \omega| < a(s)\}, \quad \omega \in \mathbb{S}^{n-1}$,
  where $a(s)$ is chosen such that $\gamma(D_{\omega,s}) = \phi(s)$

The asymptotic behavior for s going to $+\infty$

$$a(s) = s + \frac{\ln 2}{s} + o(1/s).$$

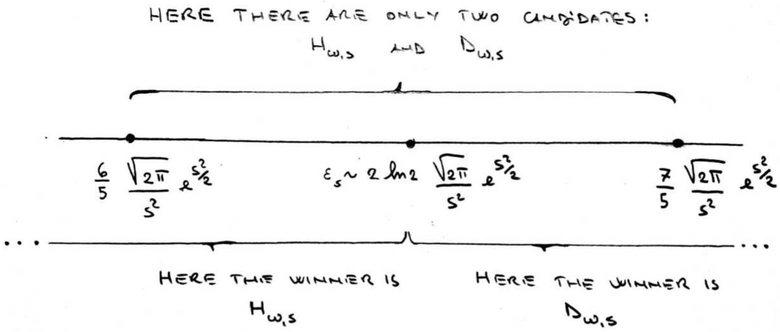
New result

Theorem (B-Julin (2018))

There exists $s_0 > 0$ such that the following holds: when $s \geq s_0$ there is a threshold ε_s such that for $\varepsilon \in [0, \varepsilon_s)$ the minimizer of $\mathcal{F}$ under volume constraint $\gamma(E) = \phi(s)$ is the half-space $H_{\omega,s}$, while for $\varepsilon \in (\varepsilon_s, \infty)$ the minimizer is the symmetric strip $D_{\omega,s}$.

ε_s is the unique value of ε for which $\mathcal{F}(H_{\omega,s}) = \mathcal{F}(D_{\omega,s})$. The asymptotic behavior is

$$\varepsilon_s = 2 \ln 2 \, \frac{\sqrt{2\pi}}{s^2} e^{\frac{s^2}{2}} \left(1 + o(1)\right) .$$


HERE THERE ARE ONLY TWO CANDIDATES:
$H_{\omega,s}$ AND $D_{\omega,s}$
$\frac{6}{5}\frac{\sqrt{2\pi}}{s^2}e^{s^2/2}$
$\varepsilon_s \sim 2\ln 2\frac{\sqrt{2\pi}}{s^2}e^{s^2/2}$
$\frac{7}{5}\frac{\sqrt{2\pi}}{s^2}e^{s^2/2}$
HERE THE WINNER IS $H_{\omega,s}$
HERE THE WINNER IS $D_{\omega,s}$

The first answer

Since symmetric sets have barycenter zero, we have the solution for the symmetric Gaussian problem (when the volume is close to one).

Corollary

There exists $s_0 > 0$ such that for $s \geq s_0$ it holds

$$P_\gamma(E) \geq 2e^{-\frac{a(s)^2}{2}} = \left(1 + \frac{\ln 2}{s^2} + o(1/s^2)\right)e^{-\frac{s^2}{2}},$$

for any symmetric set E with volume $\gamma(E) = \phi(s)$, and the equality holds if and only if $E = D_{\omega,s}$ for some $\omega \in \mathbb{S}^{n-1}$.

The second answer

We have also the optimal constant in the quantitative Gaussian isoperimetric inequality (when the volume is close to one).

Corollary

There exists $s_0 > 0$ such that for $s \geq s_0$ it holds

$$P_\gamma(E) - e^{-s^2/2} \geq c_s \beta(E),$$

for every set E with volume $\gamma(E) = \phi(s)$. The optimal constant is given by

$$c_s = \sqrt{2\pi}\, e^{s^2/2}\, \left(P_\gamma(D_{\omega,s}) - P_\gamma(H_{\omega,s})\right) = \sqrt{2\pi}\, \frac{\ln 2}{s^2} + o(1/s^2).$$

The proof:

The proof is based on a dimensional reduction.

When the vector ω is orthogonal to the barycenter, then the function ν_ω has zero average and the second variation of $\mathcal{F}$ provides the inequality

$$\int_{\partial^* E} -\nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma + \frac{\varepsilon}{\sqrt{2\pi}} \left| \int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}^{n-1}_\gamma \right|^2 \geq 0.$$

If the second term is small enough, then $\nu_\omega \equiv 0$ and E is constant in that direction. But "is it small enough?"

By using Cauchy-Schwarz inequality, we may estimate the second term by

$$\Big| \int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}_\gamma^{n-1} \Big|^2 \leq \left(\int_{\partial^* E} x_\upsilon^2 \, d\mathcal{H}_\gamma^{n-1} \right) \left(\int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}_\gamma^{n-1} \right)$$

and then, by the Eulero equation,

$$\frac{\varepsilon}{\sqrt{2\pi}} \int_{\partial^* E} x_\upsilon^2 \, d\mathcal{H}_\gamma^{n-1} \leq \frac{8}{5} \fint_{\partial^* E} \nu_\upsilon^2 \, d\mathcal{H}_\gamma^{n-1}$$

Ops, it is larger than one! :(

And we cannot shrink ε.

By using Cauchy-Schwarz inequality, we may estimate the second term by

$$\Big|\int_{\partial^* E} \nu_\omega\, x\, d\mathcal{H}_\gamma^{n-1}\Big|^2 \leq \left(\int_{\partial^* E} x_\upsilon^2\, d\mathcal{H}_\gamma^{n-1}\right)\left(\int_{\partial^* E} \nu_\omega^2\, d\mathcal{H}_\gamma^{n-1}\right)$$

and then, by the Eulero equation,

$$\frac{\varepsilon}{\sqrt{2\pi}} \int_{\partial^* E} x_\upsilon^2\, d\mathcal{H}_\gamma^{n-1} \leq \frac{8}{5} \int_{\partial^* E} \nu_\upsilon^2\, d\mathcal{H}_\gamma^{n-1}$$

Ops, it is larger than one! :(
And we cannot shrink ε.

From n to 2

No panic: all fine for $n-2$ directions!

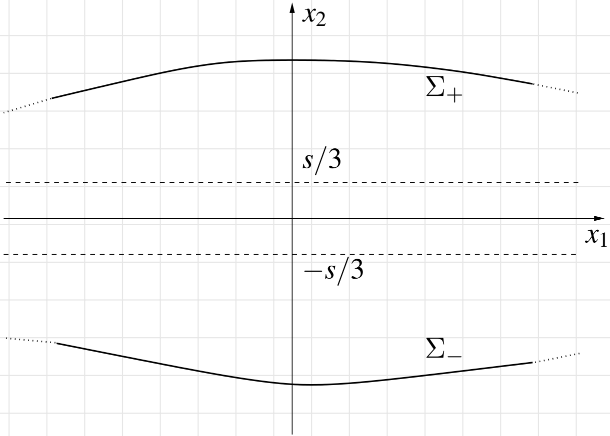
For these directions

$$\frac{\varepsilon}{\sqrt{2\pi}}\left|\int_{\partial^* E}\nu_\omega\, x\, d\mathcal{H}^{n-1}_\gamma\right|^2 \leq \frac{63}{65}\int_{\partial^* E}\nu_\omega^2\, d\mathcal{H}^{n-1}_\gamma .$$

So the problem is 2-dimensional.

From 2 to 1

A bit painfull, however...



Future, maybe...

Conjecture (1)

The solution of the symmetric problem is a cylinder $B_r^k \times \mathbb{R}^{n-k}$*, or its complement, for some* k *depending on the volume and on the dimension. Here* B_r^k *denotes the* k*-dimensional ball with radius* r*.*

Conjecture (2)

The minimizers of $\mathcal{F}$ *are symmetric for any volume (tuning* ε*). Moreover, they should be finite-dimensional (with the dimension depending on the volume).*