Floating Bodies and Flag Number

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$$K_\delta = \bigcap_{\mathrm{vol}_n(K \cap H^-) \leq \delta} H^+$$

It has been shown that [S.-Werner]

$$\lim_{\delta\to 0^+} \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_\delta)}{\delta^{\frac{2}{n+1}}}$$
$$= \frac{1}{2}\left(\frac{n+1}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n+1}} \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x),$$

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where $\kappa(x)$ is the generalized Gauß–Kronecker curvature, $\mu_{\partial K}$ denotes the surface measure on the boundary ∂K and B_2^{n-1} is the $(n-1)$-dimensional Euclidean unit ball. We put

$$\mathrm{as}(K) = \int_{\partial K} \kappa(x)^{\frac{1}{n+1}}\, d\mu_{\partial K}(x)$$

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We call the $n+1$-tuple $(\emptyset, F_0, \dots, F_{n-1}, P)$ a complete flag. The set of flags of P is denoted by $\text{flag}(P)$.

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In particular, the number of flags of P and P° are the same.

In [S.] it was shown that for polytopes

$$\lim_{\delta \to 0^+} \frac{\mathrm{vol}_n(P) - \mathrm{vol}_n(P_\delta)}{\delta \left(\ln \frac{1}{\delta}\right)^{n-1}} = \frac{|\mathrm{flag}(P)|}{n!\, n^{n-1}},$$

where $|\mathrm{flag}(P)|$ is the total number of complete flags of P.

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The affine isoperimetric inequality is

$$\frac{\text{as}(K)}{\text{as}(B_2^n)} \leq \left(\frac{\text{vol}_n(K)}{\text{vol}_n(B_2^n)}\right)^{\frac{n-1}{n+1}}$$

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The affine isoperimetric inequality gives the Blaschke-Santaló inequality

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^\circ) \leq \mathrm{vol}_n(B_2^n)^2,$$

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It cannot involve the affine surface area, since the affine surface area of a polytope is 0.

On the other hand,

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This suggests that

$$|\mathrm{flag}(P)|$$

is something like a polytopal affine surface area.

A polytopal affine isoperimetric inequality would be for arbitrary convex polytopes P

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So the above conjecture and the Mahler conjecture both hold for centrally symmetric, convex polytopes with equality for Hanner polytopes. This may be a conincidence, but I guess it is more than a coincidence.

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It follows that there is $c > 1$ such that

$$\mathrm{flag}(P) \geq c^{\sqrt{n}}$$

Let $\phi : K \to (0, \infty)$ be a continuous function and denote by Φ the measure with density ϕ, i.e., for every Borel $A \subset K$

$$\Phi(A) = \int_A \phi(x) d\lambda_n(x).$$

The *weighted floating body* is defined by

$$K_\delta^\phi = \bigcap \left\{ H^+ | \Phi(K \cap H^-) \leq \delta \right\},$$

Clearly, if $\phi \equiv 1$, then the weighted floating body is the convex floating body, i.e., $K_\delta^\phi = K_\delta$.

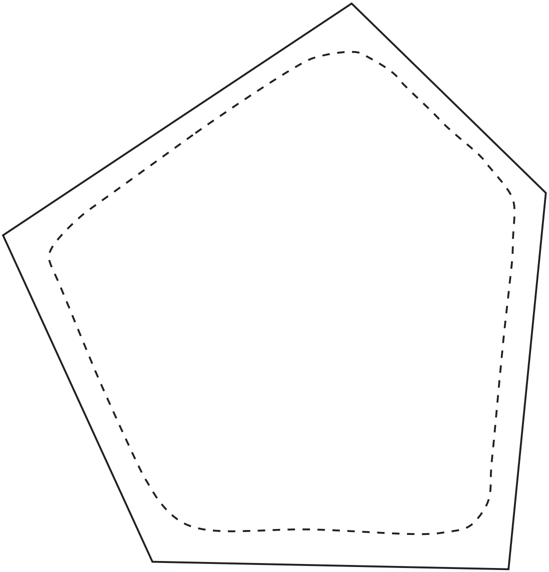
Theorem

Let P be a n-dimensional convex polytope and let $\phi, \psi : P \to (0, \infty)$ be continuous functions. Then

$$\lim_{\delta \to 0^+} \frac{\Psi(P) - \Psi(P_\delta^\phi)}{\delta \left(\ln \frac{1}{\delta}\right)^{n-1}} = \sum_{v \in \operatorname{vert} P} \frac{\psi(v)}{\phi(v)} \frac{|\operatorname{flag}_v P|}{n! \, n^{n-1}},$$

where $\operatorname{vert} P$ *is the set of vertices of P and* $|\operatorname{flag}_v P|$ *is the number of flags that have v as a vertex.*

We describe the 2-dimensional case.



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Let F be a face of P and $x \in F$. Then

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We conclude that the volume of the set $P \setminus P_\delta$ sits nearby the vertices.