Gradient Flows in Abstract Metric Spaces: Evolution Variational Inequalities and Stability

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Some preliminaries: λ-convexity and slopes

Let (X, d) be a complete metric space.

We consider a lower semicontinuous (l.s.c.) functional $\phi : X \to (-\infty, +\infty]$ with nonempty domain (i.e. ϕ is *proper* – taken for granted from now on)

$$\mathrm{Dom}(\phi) := \{x \in X : \ \phi(x) < +\infty\} .$$

Given $\lambda \in \mathbb{R}$, we say that ϕ is (geodesically) λ-convex if for every $\mathrm{x}_0, \mathrm{x}_1 \in \mathrm{Dom}(\phi)$ there exists a (minimal, constant speed) geodesic $\mathrm{x}_\vartheta : [0, 1] \to X$ such that

$$\phi(\mathrm{x}_\vartheta) \le (1-\vartheta)\phi(\mathrm{x}_0) + \vartheta\phi(\mathrm{x}_1) - \frac{\lambda}{2}\vartheta(1-\vartheta)\mathrm{d}^2(\mathrm{x}_1, \mathrm{x}_0) \qquad \forall \vartheta \in [0, 1] .$$

In particular, in this case $\mathrm{Dom}(\phi)$ is a geodesic space.

If ϕ is λ-convex, one can show that the functional $x \mapsto \phi(x) - \frac{\lambda}{2}\mathrm{d}^2(x,o)$ is linearly bounded from below for all $o \in X$:

$$\phi(x) \geq \frac{\lambda}{2}\mathrm{d}^2(x,o) - \ell_o \mathrm{d}(x,o) - c_o \qquad \forall x \in X\,, \quad \text{for some } \ell_o, c_o \geq 0\,.$$

The metric slope $|\partial\phi|$ is defined for all $x \in \mathrm{Dom}(\phi)$ by

$$|\partial\phi|(x) := \limsup_{y \to x} \frac{(\phi(x) - \phi(y))_+}{\mathrm{d}(x,y)}\,,$$

with $|\partial\phi|(x) := +\infty$ if $x \in X \setminus \mathrm{Dom}(\phi)$ and $|\partial\phi|(x) := 0$ if $x \in \mathrm{Dom}(\phi)$ is isolated.

If ϕ is λ-convex then $|\partial\phi|$ coincides with the (l.s.c.) global λ-slope:

$$\mathfrak{L}_\lambda[\phi](x) := \sup_{y \neq x} \frac{\left(\phi(x) - \phi(y) + \frac{\lambda}{2}\mathrm{d}^2(x,y)\right)_+}{\mathrm{d}(x,y)}\,.$$

EVI and Gradient Flows

First we want to give a meaning to $\dot{u} = -\partial\phi(u)$ in our metric framework.

Evolution Variational Inequalities (EVI) [Ambrosio-Gigli-Savaré '05]

A continuous curve $u \colon t \in (0, +\infty) \mapsto u_t \in \mathrm{Dom}(\phi)$ is a solution to $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$ if

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}\,\mathsf{d}^2(u_t, v) + \frac{\lambda}{2}\mathsf{d}^2(u_t, v) \le \phi(v) - \phi(u_t) \qquad \forall t > 0\,,\ \forall v \in \mathrm{Dom}(\phi)\,.$$

Here

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\,\zeta(t) := \limsup_{h\downarrow 0} \frac{\zeta(t+h) - \zeta(t)}{h} \qquad \text{(upper right Dini derivative)}\,.$$

Gradient Flows (GF)

A λ-Gradient Flow of ϕ is a family of continuous maps $\mathsf{S}_t : \overline{\mathrm{Dom}(\phi)} \to \overline{\mathrm{Dom}(\phi)}$, $t \ge 0$, such that for every $u_0 \in \overline{\mathrm{Dom}(\phi)}$ there hold

$$\mathsf{S}_{t+h}(u_0) = \mathsf{S}_h(\mathsf{S}_t(u_0)) \quad \forall t, h \ge 0\,, \qquad \lim_{t\downarrow 0} \mathsf{S}_t(u_0) = \mathsf{S}_0(u_0) = u_0\,,$$

the curve $t \mapsto \mathsf{S}_t(u_0)$ is a solution of $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$.

A classical example: Hilbert spaces

Let $(X, \langle \cdot \rangle)$ be a Hilbert space, with $\mathsf{d}(x, y) := |x - y| = \sqrt{\langle x - y, x - y \rangle}$. Let $\phi : X \to (-\infty, +\infty]$ be a l.s.c. λ-convex functional. In other words, $x \mapsto \phi(x) - \frac{\lambda}{2}|x|^2$ is a convex functional *in the usual sense*.

Then [Brézis '73] a continuous curve $u \colon t \in (0, +\infty) \mapsto u_t \in \mathrm{Dom}(\phi)$ is a solution to $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$ if and only if u is locally Lipschitz and

$$\dot{u}_t \in -\partial\phi(u_t) \qquad \text{for a.e. } t > 0$$

(for *every* $t > 0$ if we use right derivatives), where

$$w \in \partial\phi(u) \qquad \Leftrightarrow \qquad \langle w, v - u \rangle + \frac{\lambda}{2} |v - u|^2 \leq \phi(v) - \phi(u) \quad \forall v \in X \, ,$$

i.e. $\partial\phi$ is the subgradient of ϕ. In this case,

$$|\partial\phi|(u) := \min\{|w| \colon \; w \in \partial\phi(u)\} \, .$$

A more elaborate example: drift diffusion with nonlocal interaction

Let $\mathcal{X} := \mathscr{P}_2(\mathbb{R}^d)$ be the space of Borel probability measures, with finite quadratic moment, endowed with the Wasserstein distance W_2.

We consider the following functional on $\mathcal{X}$:

$$\phi(\mu) := \int_{\mathbb{R}^d} \varrho \log \varrho \, \mathrm{d}x + \int_{\mathbb{R}^d} V \, \mathrm{d}\mu + \frac{1}{2} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathcal{W}(x-y) \, \mathrm{d}\mu(y) \right) \mathrm{d}\mu(x) \qquad \text{if } \mu \equiv \varrho \mathscr{L}^d \, ,$$
$$\phi(\mu) := +\infty \qquad \text{if } \mu \not\ll \mathscr{L}^d \, ,$$

i.e. the sum of internal, potential and interaction energy. Here $V : \mathbb{R}^d \to \mathbb{R}$ is a l.s.c. convex function and $\mathcal{W} : \mathbb{R}^d \to \mathbb{R}^+$ is a $C^1(\mathbb{R}^d)$, even and convex function satisfying a suitable “doubling” condition.

Then [Carrillo-McCann-Villani ’03, Ambrosio-Gigli-Savaré ’05] the functional ϕ admits a GF in $\mathcal{X}$, which is given by solutions to the drift-diffusion (with interaction) equation

$$\partial_t \varrho_t = \Delta \varrho_t + \operatorname{div} \left[\varrho_t \left(\nabla V + \nabla \mathcal{W} * \varrho_t \right) \right] \quad \text{in } \mathbb{R}^d \, , \qquad \lim_{t \to 0} \varrho_t \, \mathscr{L}^d = \mu_0 \quad \text{in } \mathscr{P}_2(\mathbb{R}^d) \, .$$

Main properties of solutions to EVI

Theorem

Let $\phi : X \to (-\infty, +\infty]$ be a l.s.c. functional and $\lambda \in \mathbb{R}$. Let $u, u^1, u^2 \in C^0([0, +\infty); X)$ be solutions to $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$. The following properties hold:

- *λ-contraction and uniqueness:*

$$\mathsf{d}(u_t^1, u_t^2) \le \mathrm{e}^{-\lambda(t-s)} \mathsf{d}(u_s^1, u_s^2) \qquad \forall\, 0 \le s < t < +\infty\,.$$

  In particular, for each $u_0 \in \overline{\mathrm{Dom}(\phi)}$ there is at most one solution s.t. $\lim_{t \downarrow 0} u_t = u_0$.

- *Regularizing effects:*
  - *u is locally Lipschitz in $(0, +\infty)$ and $u_t \in \mathrm{Dom}(|\partial\phi|) \subset \mathrm{Dom}(\phi)$ for all $t > 0$;*
  - *the map $t \in [0, +\infty) \mapsto \phi(u_t)$ is nonincreasing and (locally) semi-convex;*
  - *the map $t \in [0, +\infty) \mapsto \mathrm{e}^{\lambda t} |\partial\phi|(u_t)$ is nonincreasing and right continuous.*

- *A priori estimates: for every $v \in \mathrm{Dom}(\phi)$ and $t > 0$*

$$\frac{\mathrm{e}^{\lambda t}}{2} \mathsf{d}^2(u_t, v) + \mathsf{E}_\lambda(t) \left(\phi(u_t) - \phi(v)\right) + \frac{(\mathsf{E}_\lambda(t))^2}{2} |\partial\phi|^2(u_t) \le \frac{1}{2} \mathsf{d}^2(u_0, v)\,,$$

  where $\mathsf{E}_\lambda(t) := \int_0^t e^{\lambda s}\, \mathrm{d}s$.

Theorem (continued)

- *Right, left limits and energy identity:* *for every $t > 0$ the right limits*

$$|\dot{u}_{t+}| := \lim_{h \downarrow 0} \frac{\mathrm{d}(u_{t+h}, u_t)}{h}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\phi(u_{t+}) := \lim_{h \downarrow 0} \frac{\phi(u_{t+h}) - \phi(u_t)}{h}$$

exist finite, satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(u_{t+}) = -|\dot{u}_{t+}|^2 = -|\partial\phi|^2(u_t) = -\mathfrak{L}_\lambda^2[\phi](u_t) \qquad \forall t > 0$$

and define a right-continuous map. In particular, the functional $x \mapsto \phi(x) - \frac{\lambda}{2}\mathrm{d}^2(x, o)$ *is linearly bounded from below for all* $o \in X$.

Minimizing Movements (MM)

Given $\tau > 0$, we consider the *quadratically-perturbed* functional

$$\Phi(\tau, U, V) := \frac{1}{2\tau} \mathrm{d}^2(U, V) + \phi(V) \qquad \forall U, V \in X\,.$$

We say that $\{U_\tau^n\}_{n\in\mathbb{N}}$ is a discrete minimizing sequence if

$$U_\tau^n \in \operatorname*{Argmin}_{V\in X} \Phi(\tau, U_\tau^{n-1}, V) \qquad \forall n \in \mathbb{N} \setminus \{0\}\,,$$

i.e. U_τ^n satisfies

$$\frac{1}{2\tau} \mathrm{d}^2(U_\tau^{n-1}, U_\tau^n) + \phi(U_\tau^n) \leq \frac{1}{2\tau} \mathrm{d}^2(U_\tau^{n-1}, V) + \phi(V) \qquad \forall V \in X\,.$$

The corresponding discrete minimizing movement is the piecewise-constant interpolant

$$\overline{U}_\tau(t) := U_\tau^n \qquad \text{if } t \in ((n-1)\tau, n\tau]\,, \qquad \overline{U}_\tau(0) = U_\tau^0 \approx u_0\,.$$

Following [De Giorgi '93, Almgren-Taylor-Wang '93, Jordan-Kinderlehrer-Otto '98], the MM method can be used to *construct* the gradient flow of ϕ. However, without coercivity assumptions on ϕ, one cannot hope to have *exact* minimizers.

Ekeland's variational principle and relaxed MM

Ekeland's variational principle

Let $\Phi : X \to (-\infty, +\infty]$ be a l.s.c. functional bounded from below. Then for every $U \in \mathrm{Dom}(\Phi)$ and every $\eta > 0$ there exists $U_\eta \in \mathrm{Dom}(\Phi)$ s.t.

$$\begin{aligned} &\Phi(U_\eta) \le \Phi(U) - \eta\, \mathsf{d}(U_\eta, U) \\ &\Phi(U_\eta) < \Phi(V) + \eta\, \mathsf{d}(U_\eta, V) \qquad \text{for every } V \in X \setminus \{U_\eta\}\,. \end{aligned}$$

In particular,

$$|\partial\Phi|(U_\eta) \le \mathfrak{L}_0[\Phi](U_\eta) \le \eta\,.$$

Our idea is to apply Ekeland's variational principle to the functional

$$V \mapsto \Phi(\tau, U^{n-1}_{\tau,\eta}, V) = \frac{1}{2\tau}\mathsf{d}^2(U^{n-1}_{\tau,\eta}, V) + \phi(V)\,.$$

By letting $U \equiv U^{n-1}_{\tau,\eta}$ and choosing the above η carefully, we can find $U^n_{\tau,\eta}$ satisfying

$$\frac{1}{2\tau}\mathsf{d}^2(U^{n-1}_{\tau,\eta}, U^n_{\tau,\eta}) + \phi(U^n_{\tau,\eta}) \le \frac{1}{2\tau}\mathsf{d}^2(U^{n-1}_{\tau,\eta}, V) + \phi(V) + \frac{\eta}{2}\,\mathsf{d}(U^{n-1}_{\tau,\eta}, U^n_{\tau,\eta})\,\mathsf{d}(U^n_{\tau,\eta}, V)$$

for every $V \in X$ and

$$\frac{1}{2\tau}\mathsf{d}^2(U^{n-1}_{\tau,\eta}, U^n_{\tau}) + \phi(U^n_{\tau,\eta}) \le \phi(U^{n-1}_{\tau,\eta})\,.$$

Two key inequalities satisfied by η-Ekeland movements

We denote by $\overline{U}_{\tau,\eta}$ the piecewise-constant interpolant of the η-Ekeland sequence $\{U^n_{\tau,\eta}\}_{n\in\mathbb{N}}$, which we call a discrete η-Ekeland movement.

In order to generate such a movement, we only need ϕ to be a l.s.c. functional quadratically bounded from below (τ small enough).

Let $\phi : X \to (-\infty, +\infty]$ be a l.s.c. λ-convex functional ($\lambda \le 0$). Then, for any η-Ekeland sequence $\{U^n_{\tau,\eta}\}$ there hold

$$\tau \left(1 - \tfrac{\eta}{2}\tau\right)^2 |\partial\phi|^2(U^n_{\tau,\eta}) \le \frac{\mathrm{d}^2(U^{n-1}_{\tau,\eta}, U^n_{\tau,\eta})}{\tau}$$

and

$$\left(1 - \tfrac{\eta-\lambda}{2}\tau\right) \frac{\mathrm{d}^2(U^{n-1}_{\tau,\eta}, U^n_{\tau,\eta})}{\tau} \le \phi(U^{n-1}_{\tau,\eta}) - \phi(U^n_{\tau,\eta})\,.$$

Such inequalities are closely related to the energy identity satisfied by solutions to EVI.

Uniform discrete-approximation error estimates

By exploiting the above inequalities plus the EVI properties, we can prove the following.

Theorem

Let $\phi : X \to (-\infty, +\infty]$ be a l.s.c. λ-convex functional ($\lambda \leq 0$), which admits a λ-Gradient Flow. Fix a time interval $[0, T]$ and $\tau \in (0, T)$. Then, if $U^0_{\tau,\eta} = u_0 \in \mathrm{Dom}(|\partial\phi|)$, there exists a constant $C = C(T, \lambda, \eta) > 0$ such that

$$\mathrm{d}(u_t, \overline{U}_{\tau,\eta}(t)) \leq C\, |\partial\phi|(u_0)\, \sqrt{\tau} \qquad \forall t \in [0, T]\,,$$

whence $\overline{U}_{\tau,\eta}(t) \to u_t$ as $\tau \downarrow 0$ with rate $\sqrt{\tau}$ (at least).

Thus, the minimizing movement (limit of $\overline{U}_{\tau,\eta}(t)$ as $\tau \downarrow 0$) exists and coincides with u_t.

The stability problem

We consider the delicate problem of stability w.r.t. ϕ.

That is, let $\phi^h : X \to (-\infty, +\infty]$, $h \in \mathbb{N}$, be a family of l.s.c. functionals “converging” in a suitable sense as $h \to \infty$ to a l.s.c. functional $\phi : X \to (-\infty, +\infty]$.

We suppose that each ϕ^h admits a λ-Gradient Flow S^h (except ϕ).

The crucial questions

(Under which assumptions) Can we deduce that

also ϕ admits a λ-Gradient Flow S

and that

$\mathsf{S}^h_t(u^h_0)$ converges to $\mathsf{S}_t(u_0)$ as $h \to \infty$, if $u^h_0 \to u_0$?

Γ and Mosco convergence

Having in mind the Hilbert case, natural assumptions involve Γ-convergence [Dal Maso '93]. We recall the definitions of Γ- lim inf and Γ- lim sup of a sequence $\{\phi^h\}_{h\in\mathbb{N}}$:

$$\Gamma\text{-}\liminf_{h\to\infty} \phi^h(x) := \inf\left\{ \liminf_{h\to\infty} \phi^h(x^h) : \; x^h \to x \right\} = \lim_{r\downarrow 0} \liminf_{h\to\infty} \inf_{B_r(x)} \phi^h ,$$

$$\Gamma\text{-}\limsup_{h\to\infty} \phi^h(x) := \inf\left\{ \limsup_{h\to\infty} \phi^h(x^h) : \; x^h \to x \right\} = \lim_{r\downarrow 0} \limsup_{h\to\infty} \inf_{B_r(x)} \phi^h ,$$

for all $x \in X$. If the Γ- lim inf and the Γ- lim sup coincide, we set

$$\phi = \Gamma\text{-}\lim_{h\to\infty} \phi^h = \Gamma\text{-}\liminf_{h\to\infty} \phi^h = \Gamma\text{-}\limsup_{h\to\infty} \phi^h ,$$

in which case we say that $\{\phi^h\}$ Γ-converges to ϕ. This is equivalent to

$$\begin{array}{|llr|} \forall x \in X, \; x^h \to x & \Rightarrow \quad \liminf_{h\to\infty} \phi^h(x^h) \ge \phi(x) & (*) \\ \forall x \in X \;\; \exists \{x^h\} : & x^h \to x , \quad \phi^h(x^h) \to \phi(x) . & \end{array}$$

If X is Hilbert one also has weak topology. We say that $\{\phi^h\}$ Mosco-converges to ϕ if it Γ-converges w.r.t. both the strong and the weak topology, i.e. $(*)$ holds for all $x^h \rightharpoonup x$.

The stability result in the Hilbert case

Theorem (Crandall, Liggett, Bénilan, Pazy, Attouch – mostly during the 70's)

Let X be a Hilbert space and $\{\phi^h\}_{h\in\mathbb{N}} \cup \{\phi\}$ be a sequence of l.s.c. and convex functionals. Let $A^h := \partial\phi^h$ and $A := \partial\phi$. Then the following properties are equivalent:

- *Convergence of the flows:* if $u_0^h \to u_0 \in \mathrm{Dom}(\phi)$, with $u_0^h \in \mathrm{Dom}(\phi^h)$,

$$\lim_{h\to\infty} \mathsf{S}_t^h(u_0^h) = \mathsf{S}_t(u_0) \qquad \forall t \geq 0\,.$$

- *Convergence of the resolvents:* for every $u \in X$ and $\tau > 0$

$$\lim_{h\to\infty} (I + \tau A^h)^{-1} u = (I + \tau A)^{-1} u\,.$$

- *Convergence of the Moreau-Yosida regularizations:* for every $u \in X$ and $\tau > 0$

$$\lim_{h\to\infty} \inf_{v\in X} \phi^h(v) + \frac{1}{2\tau}\mathsf{d}^2(v,u) = \inf_{v\in X} \phi(v) + \frac{1}{2\tau}\mathsf{d}^2(v,u)\,.$$

- *Mosco-convergence of the functionals:* $\{\phi^h\}$ Mosco-converges to ϕ.
- *G-convergence of the subgradients:* for every $v \in A(u)$ there exist $\{u^h\}, \{v^h\}$ s.t.

$$v^h \in A^h u^h\,, \quad u^h \to u\,, \quad v^h \to v\,.$$

Some related remarks

- Mosco-limits of convex functionals are convex: in particular, S exists thanks e.g. to the Crandall-Liggett Theorem (without assuming *a priori* the convexity of ϕ).
- For every $v \in A(u)$ one can construct a recovery sequence $v^h \in A^h(u^h)$ s.t.

$$u^h \to u\,, \quad v^h \to v\,, \quad \phi^h(u^h) \to \phi(u)\,.$$

- If $\{\phi^h\}$ is strongly coercive (bdd sequences $\{x^h\}$ s.t. $\phi^h(x^h) \le C$ are rel. compact), then Mosco convergence $\Leftrightarrow$ Γ-convergence. Otherwise, limits of $\phi^h(x^h)$ along weakly convergent sequences are involved, whence the weak Γ-lim inf.
- The resolvent operator is strictly related to MM:

$$U_\tau^{n,h} = (1 + \tau A^h)^{-1}\, U_\tau^{n-1,h}\,.$$

- In order to prove convergence of the flows, it is therefore convenient to exploit convergence of the minimizing movements along with uniform error estimates:

$$\mathrm{d}(u_t^h, u_t) \le \mathrm{d}(u_t^h, \overline{U}_\tau^h(t)) + \mathrm{d}(\overline{U}_\tau^h(t), \overline{U}_\tau(t)) + \mathrm{d}(\overline{U}_\tau(t), u_t)\,,$$

where $u_t^h := \mathrm{S}_t^h(u_0^h)$ and $u_t := \mathrm{S}_t(u_0)$.

Additional difficulties due to the abstract metric setting

- We do not know a priori whether the limit λ-Gradient Flow S exists.
- Resolvents are not well defined: one should use η-Ekeland movements instead.
- A natural weak topology is missing.
- We would like to study stability without strong-coercivity assumptions.
- On the other hand, if we lack coercivity, minimizing movements (a fortiori η-Ekeland movements) are not stable under Γ-convergence.

We point out that, at least in the strongly coercive case, it is possible to pass to the limit in the integral version of the EVI:

$$\frac{e^{\lambda(t-s)}}{2}\, d^2(u_t^h, v^h) - \frac{1}{2} d^2(u_s^h, v^h) \leq \mathsf{E}_\lambda(t-s) \left(\phi^h(v^h) - \phi(u_t^h) \right),$$

for every $0 \leq s \leq t$ and $v^h \in \mathrm{Dom}(\phi^h)$, which yields existence of S “for free”.

The main stability result

Theorem

Let $\{\phi^h\}_{h\in\mathbb{N}} \cup \{\phi\}$ be a sequence of l.s.c. functionals. Let each ϕ^h admit a λ-Gradient Flow S^h and let ϕ be λ-convex. The following claims are equivalent:

Convergence of the flows: also S exists and if $u_0^h \to u_0 \in \overline{\mathrm{Dom}(\phi^\infty)}$, $u_0^h \in \overline{\mathrm{Dom}(\phi^h)}$,

$$\lim_{h\to\infty} \mathsf{S}_t^h(u_0^h) = \mathsf{S}_t(u_0) \qquad \forall t \geq 0\,.$$

Recovery sequence: for every $u \in \mathrm{Dom}(|\partial\phi|)$ there exists $u^h \in \mathrm{Dom}(|\partial\phi^h|)$ s.t.

$$u^h \to u\,, \quad \phi^h(u^h) \to \phi(u)\,, \quad \limsup_{h\to\infty} |\partial\phi^h|(u^h) \leq |\partial\phi|(u)\,.$$

Γ-convergence of ϕ^h and $|\partial\phi^h|$: $\phi = \Gamma\text{-}\lim\, \phi^h$ and $|\partial\phi| = \Gamma\text{-}\lim\, |\partial\phi^h|$ in $\overline{\mathrm{Dom}(\phi)}$.

Qualified Γ-convergence: $\Gamma\text{-}\limsup\, \phi^h \leq \phi$ in $\mathrm{Dom}(|\partial\phi|)$ and for every $u \in \mathrm{Dom}(|\partial\phi|)$, $\varepsilon > 0$ and $\bar\tau > 0$, there exists $\tau \in (0,\bar\tau)$ s.t.

$$\liminf_{h\to\infty} \inf_{B_\tau(u)} \phi^h \geq \inf_{B_\tau(u)} \phi - \varepsilon\tau\,.$$

Local Moreau-Yosida regularizations: $\Gamma\text{-}\limsup\, \phi^h \leq \phi$ in $\mathrm{Dom}(|\partial\phi|)$ and for every $u \in \mathrm{Dom}(|\partial\phi|)$, $\varepsilon > 0$ and $\bar\tau > 0$, there exists $\tau \in (0,\bar\tau)$ s.t.

$$\liminf_{h\to\infty} \inf_{v\in X} \phi^h(v) + \frac{1}{2\tau}\mathrm{d}^2(v,u) \geq \inf_{v\in X} \phi(v) + \frac{1}{2\tau}\mathrm{d}^2(v,u) - \varepsilon\tau\,.$$

Strategy of proof of the existence of the limit flow

- We generate a η-Ekeland sequence $\{U^n_{\tau,\eta}\}$ for ϕ, which satisfies

$$\tau\left(1-\tfrac{\eta}{2}\tau\right)^2 |\partial\phi|^2(U^n_{\tau,\eta}) \le \frac{\mathrm{d}^2(U^{n-1}_{\tau,\eta},U^n_{\tau,\eta})}{\tau} \le \frac{\phi(U^{n-1}_{\tau,\eta})-\phi(U^n_{\tau,\eta})}{1-\frac{\eta-\lambda}{2}\tau}\,. \qquad (\star)$$

- We exploit Γ-convergence of ϕ^h and $|\partial\phi^h|$ to approximate $U^n_{\tau,\eta}$ by sequences $U^{n,h}_{\tau,\eta}$ satisfying, for large h, the ε-version of $(\star)$:

$$\lim_{h\to\infty}\sup_{t\in[0,T]}\mathrm{d}(\overline{U}_{\tau,\eta}(t),\overline{U}^h_{\tau,\eta}(t))=0\,.$$

- We use the discrete-approximation error estimate, which yields

$$\mathrm{d}(u^h_t,\overline{U}^h_{\tau,\eta}(t)) \le C\left(|\partial\phi^h|(u^h_0)\sqrt{\tau}+\sqrt{\varepsilon/\tau}\right) \qquad \forall t\in[0,T]\,.$$

- By combining the two estimates and choosing $U^{0,h}_{\tau,\eta}$ appropriately, we deduce that

$$\limsup_{h,k\to\infty}\sup_{t\in[0,T]}\mathrm{d}(u^h_t,u^k_t) \le C'\left(\sqrt{\tau}+\sqrt{\varepsilon/\tau}\right),$$

which shows that $\{u^t_h\}_h$ is Cauchy, since $\tau>0$ and $\varepsilon>0$ are arbitrary.

An application to RCD spaces

Let $(X, \mathsf{d}, \boldsymbol{m})$ be an $\mathrm{RCD}(\lambda, \infty)$ metric measure space and let $\psi : X \to (-\infty, +\infty]$ be a continuous and geodesically λ-convex functional.

Theorem (Sturm '14)

If (X, d) is locally compact then ψ admits a λ-Gradient Flow.

Corollary of our results

The local-compactness assumption can be removed.

Indeed, Sturm's proof relies on the construction of the λ-GF for the functional

$$\phi(\mu) := \int_X \psi \, \mathrm{d}\mu \qquad \text{in } (\mathscr{P}_2(X), W_2)$$

by means of the approximations $\phi^h(\mu) := \phi(\mu) + \frac{1}{h}\,\mathrm{Ent}(\mu|\boldsymbol{m})$. At least when $\boldsymbol{m} \in \mathscr{P}(X)$, one can check that the assumptions of our main stability result are met.

Some extensions concerning the stability result

- Completeness of X can be dropped: we only need ϕ to have complete sublevels.
- Convexity of ϕ can, to some extent, be relaxed: if $\mathrm{Dom}(\phi)$ is geodesic, then it is just a consequence of the existence of the flows for ϕ^h.
- Alternatively, it is enough to ask that ϕ is approximately λ-convex, namely that for every $x_0, x_1 \in \mathrm{Dom}(\phi)$ and every $\vartheta, \varepsilon \in (0,1)$ there exists $x_{\vartheta,\varepsilon} \in \mathrm{Dom}(\phi)$ s.t.

$$\phi(x_{\vartheta,\varepsilon}) \leq (1-\vartheta)\phi(x_0) + \vartheta\phi(x_1) - \frac{\lambda-\varepsilon}{2}\vartheta(1-\vartheta)d^2(x_1,x_0)$$

and

$$d(x_{\vartheta,\varepsilon},x_0) \leq \vartheta d(x_1,x_0) + \varepsilon\,, \qquad d(x_\vartheta,x_1) \leq (1-\vartheta)d(x_1,x_0) + \varepsilon\,.$$

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- Convexity of ϕ can, to some extent, be relaxed: if $\mathrm{Dom}(\phi)$ is geodesic, then it is just a consequence of the existence of the flows for ϕ^h.
- Alternatively, it is enough to ask that ϕ is approximately λ-convex, namely that for every $x_0, x_1 \in \mathrm{Dom}(\phi)$ and every $\vartheta, \varepsilon \in (0,1)$ there exists $x_{\vartheta,\varepsilon} \in \mathrm{Dom}(\phi)$ s.t.

$$\phi(x_{\vartheta,\varepsilon}) \le (1-\vartheta)\phi(x_0) + \vartheta\phi(x_1) - \frac{\lambda-\varepsilon}{2}\vartheta(1-\vartheta)d^2(x_1,x_0)$$

and

$$d(x_{\vartheta,\varepsilon},x_0) \le \vartheta d(x_1,x_0) + \varepsilon\,, \qquad d(x_\vartheta,x_1) \le (1-\vartheta)d(x_1,x_0) + \varepsilon\,.$$

THANK YOU FOR YOUR ATTENTION!