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Large-time behavior in hypocoercive BGK-models

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Decay for nonsymmetric ODEs: find Lyapunov functionals!

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{R}^n \tag{1}$$

Definition: $\mathbf{C}$ is *coercive* if $x^T\mathbf{C}x \geq \kappa\|x\|^2\ \forall x$ (for some $\kappa > 0$).

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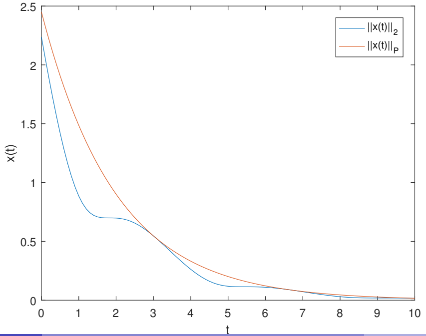
ex: $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\lambda_{\mathbf{C}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \Rightarrow$ decay rate $= \frac{1}{2}$ for (1).

- $\mathbf{C}$ not coercive $\Rightarrow$ no decay of $\|x(t)\|_2$ by trivial energy method!
- But decay of modified norm $\|x(t)\|_{\mathbf{P}} := \sqrt{x^T\,\mathbf{P}\,x}$; $\mathbf{P} := [2\ -1; -1\ 2]$



Decay for nonsymmetric ODEs: find Lyapunov functionals!

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{R}^n$$

Definition: $\mathbf{C}$ is *hypocoercive* if $\exists\, \mu > 0$ such that:

$$\Re(\lambda_j) \geq \mu, \qquad j = 1, \ldots n.$$

If all eigenvalues of $\mathbf{C}$ are non-defective:

$$\exists c \geq 1 : \qquad \|x(t)\|_2 \leq c\|x(0)\|_2 e^{-\mu t}, \quad t \geq 0.$$

- always: $\kappa \leq \mu$

Choice of $\mathbf{P}$ / Lyapunov's direct method

Lemma 1

Let $\mathbf{C} \in \mathbb{R}^{n\times n}$ *be positive stable, i.e.* $\mu := \min\{\Re\,\lambda_{\mathbf{C}}\} > 0\,.$

1. *If all* $\lambda_{\mathbf{C}}^{\min} \in \{\lambda \in \sigma(\mathbf{C})\,|\,\Re\lambda = \mu\}$ *are non-defective*
   (i.e. geometric = algebraic multiplicity)
   $\Rightarrow \quad \exists\,\mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0\ :\quad \mathbf{PC} + \mathbf{C}^\top\mathbf{P} \geq 2\mu\mathbf{P}\,.$

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2. *If (at least) one* $\lambda_{\mathbf{C}}^{\min}$ *is defective* $\Rightarrow$
   $\forall\, \varepsilon > 0 \quad \exists\, \mathbf{P} = \mathbf{P}(\varepsilon) > 0 \ : \quad \mathbf{PC} + \mathbf{C}^\top \mathbf{P} \geq 2(\mu - \varepsilon)\mathbf{P}$.

Choice of $\mathbf{P}$ / Lyapunov's direct method

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Proof: $\mathbf{P}$ can be constructed explicitly; e.g. for $\mathbf{C}$ non-defective / diagonalizable:

$$\mathbf{P} := \sum_{j=1}^{n} z_j \otimes \bar{z}_j^\top \ ; \qquad z_j \ \ldots \text{ eigenvectors of } \mathbf{C}^\top$$

□

- $\mathbf{P}$ not unique; but the decay rates μ (or $\mu - \varepsilon$) are independent of $\mathbf{P}$.
- For complex $\mathbf{C}$: $\mathbf{P} > 0$ Hermitian with $\mathbf{PC} + \mathbf{C}^*\mathbf{P} \geq 2\mu\mathbf{P}$.

Decay of **P**–norm

- Sharp decay estimate for $\dot{x} = -\mathbf{C}x$ (non-defective case):

Let $\|x\|_P^2 := x^T \, \mathbf{P} \, x$.

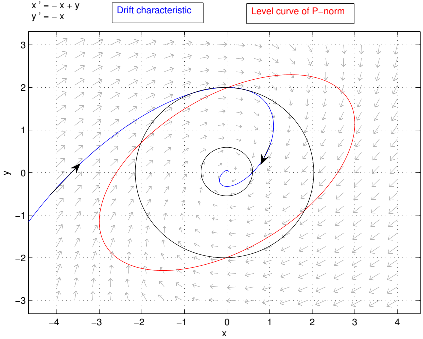
$$\frac{d}{dt}\,\|x\|_{\mathbf{P}}^2 \;=\; -x^T\big(\underbrace{\mathbf{PC} + \mathbf{C}^T\mathbf{P}}_{\geq 2\mu\mathbf{P}}\big)x \leq -2\mu\|x\|_{\mathbf{P}}^2$$

$$\Rightarrow \quad \|x(t)\|_{\mathbf{P}} \;\leq\; \|x(0)\|_{\mathbf{P}}\, e^{-\mu t}, \quad t \geq 0.$$

Decay of P–norm (cont'd)

<u>ex:</u> $\dot{x} = -\mathbf{C}x$ with $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

- At x_2-axis: trajectory $x(t)$ tangent to level curve of $|x|$:



- level curve of "distorted" vector norm $\sqrt{x^T \mathbf{P}\, x}$; $\mathbf{P} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

  $\rightarrow$ uniform decay with sharp rate $\frac{1}{2}$

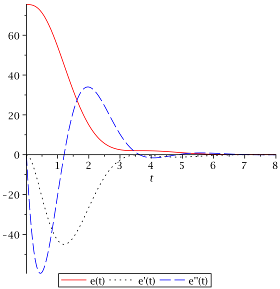
Wavy decay of energy/entropy in degenerate kinetic eq.

kinetic Fokker-Planck equation from plasma physics, for density $f(x, v, t)$, $x, v \in \mathbb{R}^d$:

$$f_t + \underbrace{v \cdot \nabla_x f}_{\text{free transport}} - \underbrace{\nabla_x V \cdot \nabla_v f}_{\text{influence of potential } V(x)} = \underbrace{\Delta_v f}_{\text{diffusion}} + \underbrace{\mathrm{div}_v(vf)}_{\text{friction}}$$

in 1D, $V(x) = x^2/2$:

$e(t) = \|f(t) - f_\infty\|^2_{L^2(f_\infty^{-1})}$



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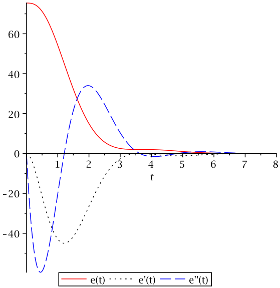

- local-in-x equilibration of $f(x, v, t)$ towards $\rho(x)\, \frac{1}{\sqrt{2\pi}} e^{-|v|^2/2}$
- (Hamiltonian) rotation in phase space "mixes" positions
  $\Rightarrow$ global equilibration

Space-inhomogen. Bhatnagar-Gross-Krook model (1954)

- application: gas dynamics, computational fluid dynamics
- simplified kinetic model: relaxation operator (instead of $Q(f,f)$):

$$f_t + v \cdot \nabla_x f = M_f(x,v,t) - f(x,v,t) \,, \quad t \geq 0 \,,\ x \in \mathbb{T}^d \,,\ v \in \mathbb{R}^d$$

- relaxation towards local Maxwellian M_f with same hydrodynamic moments as f:

$$M_f(x,v) = \frac{\rho(x)}{(2\pi T(x))^{\frac{d}{2}}}\, e^{-\frac{|v-u(x)|^2}{2T(x)}} \,,$$

$$\text{density} \quad \rho(x) := \int_{\mathbb{R}^d} f(x,v)\, dv \,,$$

$$\text{mean velocity} \quad u(x) := \frac{1}{\rho(x)} \int_{\mathbb{R}^d} v f(x,v)\, dv \,,$$

$$\text{temperature} \quad T(x) := \frac{1}{d\rho(x)} \int_{\mathbb{R}^d} |v-u(x)|^2 f(x,v)\, dv \,.$$

nonlinear BGK model

Results:

- Perthame 1989: global existence for $(x, v) \in \mathbb{R}^{2d}$ (compactness of M_f in L^1)
- Perthame-Pulvirenti 1993: uniqueness for $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$ (contraction, $\|f\|_\infty$–bounds)

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Goals:

- exponential convergence to equilibrium
- partially with sharp rates

Outline:

1. hypocoercivity
2. linear BGK models (discrete & continuous velocity) with 1 conserved quantity
3. linearized & nonlinear BGK model with 2 conserved quantities

1D linear BGK model for $f(x, v, t)$: only mass conservation

$$f_t + vf_x = Qf := M_T(v) \int_{\mathbb{R}} f(x, v, t)\, dv - f(x, v, t), \quad t \geq 0,$$

$$(x, v) \in \mathbb{T}^1 \times \mathbb{R}$$

$$f^\infty(x, v) = M_T(v) := (2\pi T)^{-1/2} e^{-v^2/2T}; \quad \text{some fixed temperature } T,$$

$$Lf := -vf_x + Qf$$

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Q ... selfadjoint on $H := L^2(\mathbb{T}^1 \times \mathbb{R}; M_T^{-1}(v)dv)$

- collision operator Q drives f to a *local* Maxwellian $\rho(x)\,M_T(v)$.
- transport operator $-v\partial_x$ leads to uniformity in x $\quad\rightarrow\quad$ hypocoercive.

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$-L$ is *not* coercive, i.e.:

$$\langle -Lf, f\rangle_H \ge \lambda \|f\|^2_{L^2(M_T^{-1})} \quad \forall f \in \{f^\infty\}^\perp$$

only holds for $\lambda = 0$; see $f = \rho(x) M_T(v) \Rightarrow$ exponential decay not obvious.

Hypocoercivity

Definition 1 (Villani 2009)

Consider L on Hilbert space H with $\mathcal{K} = \ker L$; let $\tilde{H} \hookrightarrow \mathcal{K}^\perp$ (densely)
(e.g. H ... weighted L^2, $\tilde{H}$ ... weighted H^1).
$-L$ is called hypocoercive on $\tilde{H}$ if $\exists\, \lambda > 0,\ c \geq 1$:

$$\|e^{Lt} f\|_{\tilde{H}} \leq c\, e^{-\lambda t} \|f\|_{\tilde{H}} \qquad \forall\, f \in \tilde{H}$$

- typically $c > 1$

2 velocity BGK model (= Goldstein-Taylor model)

for $f(x,t) = \binom{f_+(x,t)}{f_-(x,t)}$:

$$\partial_t f_\pm \pm \partial_x f_\pm = \pm \frac{1}{2}(f_- - f_+), \quad t \geq 0, \quad 2\pi\text{–periodic in } x$$

$f^\infty(x) = \binom{f_+^\infty}{f_-^\infty} = \binom{\frac{1}{2}}{\frac{1}{2}}$

x-Fourier modes; discrete velocity basis $\{\binom{1}{1}, \binom{1}{-1}\}$
$\Rightarrow \quad u_k(t) \in \mathbb{C}^2, \quad k \in \mathbb{Z}$

Ref's:
[Dolbeault-Mouhot-Schmeiser '15] exp. decay (only 1 conserved quantity)
[Achleitner-AA-Carlen '16] sharp decay rate

2 velocity BGK model: decay of Fourier modes

$$\frac{\mathrm{d}}{\mathrm{d}t} u_k = -\mathbf{C}_k \, u_k, \quad \mathbf{C}_k = \begin{pmatrix} 0 & ik \\ ik & 1 \end{pmatrix}, \quad k \in \mathbb{Z}$$

$$\lambda_{\mathbf{C}_0} = 0, \, 1; \qquad \lambda_{\mathbf{C}_k} = \frac{1}{2} \pm i\sqrt{k^2 - \frac{1}{4}}, \quad k \neq 0$$

$$u_0^\infty = \binom{1}{0}; \qquad u_k^\infty = \binom{0}{0}, \quad k \neq 0$$

Decay of each mode in $\mathbf{P}_k$-norm:

$$\begin{aligned} \|u_0(t) - u_0^\infty\|_2 &\leq e^{-t} \|u_0(0) - u_0^\infty\|_2 \\ \|u_k(t)\|_{\mathbf{P}_k} &\leq e^{-t/2} \|u_k(0)\|_{\mathbf{P}_k} \quad k \neq 0 \end{aligned}$$

Lemma 1 $\Rightarrow$ $\mathbf{P}_k \mathbf{C}_k + \mathbf{C}_k^* \mathbf{P}_k \geq 2 \cdot \frac{1}{2} \mathbf{P}_k \,, \quad k \neq 0$

2 velocity BGK model: decay of sequence of modes

$$k \neq 0 : \quad \mathbf{P}_k := \begin{pmatrix} 1 & \frac{-i}{2k} \\ \frac{i}{2k} & 1 \end{pmatrix} \xrightarrow{|k|\to\infty} \mathbf{I}$$

$\Rightarrow$ "generic" Lyapunov functional:

$$e(\{u_k\}) := \sqrt{\sum_{k\in\mathbb{Z}} \|u_k\|^2_{\mathbf{P}_k}} \quad \sim \quad \left\| \begin{pmatrix} f_+(x) \\ f_-(x) \end{pmatrix} \right\|_{L^2(0,2\pi;\mathbb{R}^2)}$$

Theorem 2 (Achleitner-AA-Carlen 2016)

Let $\int_0^{2\pi} [f_+^I(x) + f_-^I(x)]\, dx = 2\pi$.

$$\Rightarrow \quad \|f(t) - f^\infty\|_{L^2(0,2\pi;\mathbb{R}^2)} \leq \sqrt{3}\, e^{-t/2} \|f^I - f^\infty\|_{L^2(0,2\pi;\mathbb{R}^2)}\,, \quad t \geq 0\,.$$

- *Decay rate* $\frac{1}{2}$ *sharp.*

- $\sqrt{3} = \mathrm{cond}(\mathbf{P}_1)$

Continuous velocity BGK model: Fourier modes

$$f_t + vf_x = M_T(v) \int_{\mathbb{R}} f(x,v,t)\,dv - f(x,v,t), \quad t \geq 0$$

• x-Fourier modes $k \in \mathbb{Z}$ decouple; Hermite function basis in v

$\Rightarrow$ consider "infinite vector" $\hat{\mathbf{f}}_k(t) \in \ell^2(\mathbb{N}_0)$ for each fixed k:

$$\partial_t \hat{\mathbf{f}}_k + ik\sqrt{T}\,\mathbf{L}_1 \hat{\mathbf{f}}_k = \mathbf{L}_2 \hat{\mathbf{f}}_k\,, \quad t \geq 0;\ k \in \mathbb{Z}\,,$$

$$\mathbf{L}_1 = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ \vdots & 0 & \sqrt{3} & \ddots \end{pmatrix}, \quad \mathbf{L}_2 = \operatorname{diag}(0, -1, -1, \cdots)$$

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- need: spectral gap of the "infinite matrix" $\mathbf{C}_k := ik\sqrt{T}\,\mathbf{L}_1 - \mathbf{L}_2$, uniform in $k \in \mathbb{Z}$.
- difficult for ∞-dimensional case

Continuous velocity BGK model: decay of Fourier modes

$\rightarrow$ approximate transformation matrices $\mathbf{P}_k$: ansatz for upper left 2×2 block:

$$\mathbf{P}_k^{2\times 2} = \begin{pmatrix} 1 & -i\alpha/k \\ i\alpha/k & 1 \end{pmatrix},$$

remaining diagonal $= 1$.

Motivation: mix 0^{th}, 1^{st} mode of $\mathbf{L}_2 = \mathrm{diag}(0, -1, \cdots)$

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- Find $\alpha \in \mathbb{R}$ to maximize $\mu > 0$ in

$$\mathbf{C}_k^* \mathbf{P}_k + \mathbf{P}_k \mathbf{C}_k - 2\mu \mathbf{P}_k \geq \mathbf{0} \quad \forall\, k \in \mathbb{Z}.$$

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Lyapunov functional:

$$e(f) := \sum_{k\in\mathbb{Z}} \left\langle (f_k(v) - M_T(v)), \tilde{\mathbf{P}}_k (f_k(v) - M_T(v)) \right\rangle_{L^2(M_T^{-1})}$$

$\tilde{\mathbf{P}}_k$ ... bounded operator on $L^2(M_T^{-1})$, represented by matrix $\mathbf{P}_k$.

Result: Exp. decay for BGK model; estimated rate off by factor 2.5 (compared to numerical result).

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1D nonlinear BGK model

$$f_t(x,v,t) + vf_x = M_f(x,v,t) - f \ ; \quad x \in \mathbb{T} \ , \ v \in \mathbb{R}$$

M_f ... local Maxwellian with same mass & "temperature" (=2^{nd} moment) as f; but mean velocity := zero:

$$M_f(v) = \frac{\rho^{3/2}}{\sqrt{2\pi P}} e^{-\frac{v^2 \rho}{2P}} \ ; \quad \rho = \int f \, dv \, , \quad P = \int v^2 f \, dv$$

1D linearized BGK model

- Linearized model around temperatur $= T$; $f = M_T + h$;
  in x Fourier / in v Hermite expansion:

$$\partial_t \hat{\mathbf{h}}_k(t) + ik\sqrt{T}\,\mathbf{L}_1 \hat{\mathbf{h}}_k(t) = \mathbf{L}_3 \hat{\mathbf{h}}_k(t)\,, \quad k \in \mathbb{Z};$$

$$\mathbf{L}_3 = \mathrm{diag}(0, -1, 0, -1, -1, \cdots)\,, \quad \mathbf{L}_1 = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ \vdots & 0 & \sqrt{3} & \ddots \end{pmatrix}.$$

- 2 conserved quantities
- $\mathbf{L}_1$ couples mode 0 to mode 1; mode 2 to mode 3 (and 1)

1D linearized BGK model - simplified Lyapunov functional

$\Rightarrow$ simplified ansatz for $\mathbf{P}_k$:

$$\begin{pmatrix} 1 & -i\alpha/k & 0 & 0 \\ i\alpha/k & 1 & 0 & 0 \\ 0 & 0 & 1 & -i\beta/2k \\ 0 & 0 & i\beta/2k & 1 \end{pmatrix} \xrightarrow{k\to\infty} \mathbf{I}$$

as its upper-left 4×4 block; rest is $\mathbf{I}$.

- $\alpha = \beta = \frac{1}{3}$ yields for some $\mu_0 > 0$, with $\mathbf{C}_k := ik\mathbf{L}_1 - \mathbf{L}_3$:

$$\mathbf{P}_k\mathbf{C}_k + \mathbf{C}_k^*\mathbf{P}_k \geq 2\mu_0\mathbf{P}_k \ , \quad k\text{–uniformly.}$$

This is the essence for the decay result:

1D linearized BGK model - exponential decay

Lyapunov functional (with $h = f - M_T$):

$$e_\gamma(f) := \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \langle h_k(v), \mathbf{P}_k h_k(v)\rangle_{L^2(M_T^{-1})} \sim \|f - M_T\|^2_{\mathcal{H}^\gamma}$$

with $\mathcal{H}^\gamma := H^\gamma(0,2\pi) \otimes L^2(\mathbb{R}; M_T^{-1})$.

Theorem 3 (Achleitner-AA-Carlen 2016)

Let $T = 1$ and $\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}}(1, v^2)\, f^I\, dvdx = (1,1)$. Then

$$e_\gamma(f(t)) \le e^{-t/25} e_\gamma(f^I) \qquad t \ge 0;\ \forall\, \gamma \ge 0.$$

Remark: Estimated rate off by factor 18 (compared to numerical result).

1D nonlinear BGK model - local exponential stability

Theorem 4 (Achleitner-AA-Carlen 2016)

Additional assumptions (over Theorem 3): Let $\gamma > \frac{1}{2}$, $\|f^I - M_1\|_{\mathcal{H}^\gamma} < \delta_\gamma$ *(=explicit constant). Then*

$$e_\gamma(f(t)) \leq e^{-t/25} e_\gamma(f^I) , \quad t \geq 0.$$

Proof: Rewrite nonlinear BGK like linearized BGK + higher order remainder

True linearized BGK model

Let $T = 1$, $d = 1$:

$$\partial_t \hat{\mathbf{h}}_k + ik\,\mathbf{L}_1 \hat{\mathbf{h}}_k = \mathbf{L}_4 \hat{\mathbf{h}}_k\,, \quad k \in \mathbb{Z};$$

$$\mathbf{L}_4 = \mathrm{diag}(0,\, 0,\, 0,\, -1,\, -1, \cdots)$$

- 3 conserved quantities (mass, mean velocity, temperature)
- iterated hypocoercive structure:
  Tridiagonal matrix $\mathbf{L}_1$ couples mass-mode to velocity-mode to energy-mode to the dissipative mode 3.

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$\Rightarrow$ simplified ansatz for $\mathbf{P}_k$ (with some $\alpha, \beta, \gamma \in \mathbb{R}$):

$$\begin{pmatrix} 1 & -i\alpha/k & 0 & 0 \\ i\alpha/k & 1 & -i\beta/k & 0 \\ 0 & i\beta/k & 1 & -i\gamma/k \\ 0 & 0 & i\gamma/k & 1 \end{pmatrix}$$

as its upper-left 4×4 block; rest is $\mathbf{I}$.

$\Rightarrow$ exp. decay (also for $d = 2, 3$)

Conclusion

- algebraic lemma $\mathbf{PC} + \mathbf{C}^*\mathbf{P} \geq 2\mu\mathbf{P}$ yields the generic decay norm in linear ODEs
- exact / approximate generalizations to kinetic BGK-models: exponential decay for discrete / continuous velocities, linearized / nonlinear

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- exact / approximate generalizations to kinetic BGK-models: exponential decay for discrete / continuous velocities, linearized / nonlinear

References

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