Global-local mixing for one-dimensional intermittent maps

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(joint work with Claudio Bonanno and Paolo Giulietti)

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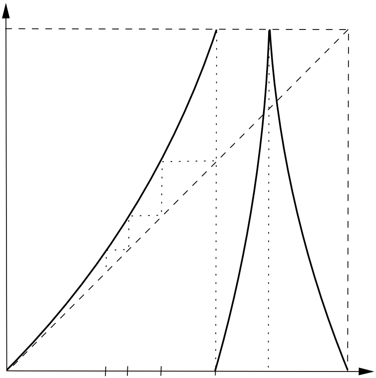
One-dimensional maps with an indifferent fixed point

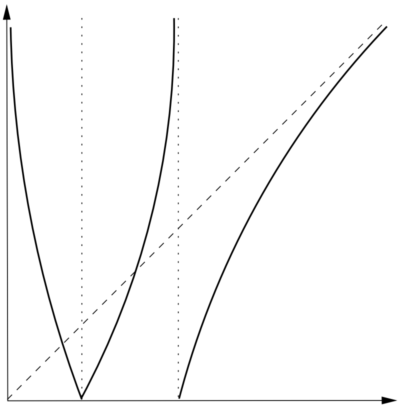
We study full-branched 1D expanding maps with an indifferent fixed point, preserving an absolutely continuous infinite measure:

Case (A): Maps $(0, 1) \longrightarrow (0, 1)$ with C^2-regular fixed point at 0



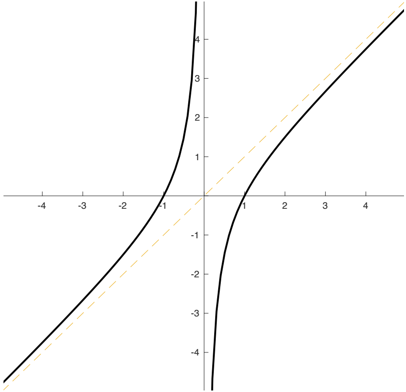
One-dimensional maps with an indifferent fixed point

Case (B): Maps $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with fixed point at $+\infty$, preserving the Lebesgue measure



One-dimensional maps with an indifferent fixed point

... and also the Boole map $T : \mathbb{R} \longrightarrow \mathbb{R}$, $T(x) := x - \dfrac{1}{x}$.



Standard assumptions for case (A):

$\exists$ partition $\{I_j\}_{j\in\mathcal{J}}$, with $I_j := (a_j, a_{j+1})$, $a_0 := 0$ and either $\mathcal{J} := \{0, 1, \ldots, N-1\}$ (in which case $a_N = 1$) or $\mathcal{J} := \mathbb{N}$ (in which case $\lim_n a_n = 1$), s.t.

(A1) $T|_{(a_j, a_{j+1})}$ has unique extension $\tau_j : [a_j, a_{j+1}] \longrightarrow [0, 1]$, twice-differentiable, bijective

(A2) $\exists \Lambda > 1$ s.t. $|\tau_j'| \geq \Lambda$, $\forall j \geq 1$

(A3) $\exists K > 0$ s.t. $\left| \dfrac{\tau_j''}{(\tau_j')^2} \right| \leq K$, $\forall j \geq 0$

(A4) τ_0 convex, $\tau_0(0) = 0$, $\tau_0'(0) = 1$, and $\tau_0'(x) > 1$, $\forall x \in (0, a_1]$

Theorem *Thaler '80–'83*

Under (A1)–(A4):

1. T preserves an infinite invariant measure μ, absolutely continuous w.r.t. m (= Lebesgue measure) and unique up to factors. Moreover, $h := \frac{d\mu}{dm} > 0$ and unbounded only near 0
2. T is conservative and exact (w.r.t. m or μ, which is the same)

Standard assumptions for case (B):

$\exists$ partition $\{I_j\}_{j\in\mathcal{J}}$, with $I_j := (a_{j+1}, a_j)$, $a_0 := +\infty$ and either $\mathcal{J} := \{0, 1, \ldots, N-1\}$ (in which case $a_N = 0$) or $\mathcal{J} := \mathbb{N}$ (in which case $\lim_n a_n = 0$), s.t.

(B1) $T|_{(a_{j+1},a_j)}$ has unique extension τ_j defined on $[a_{j+1}, a_j)$ or $(a_{j+1}, a_j]$, twice-differentiable, bijective onto $\mathbb{R}^+$

(B2) $\exists \Lambda > 1$ s.t. $|\tau_j'| \geq \Lambda$, $\forall j \geq 1$

(B3) $\exists K > 0$ s.t. $\left| \frac{\tau_j''}{(\tau_j')^2} \right| \leq K$, $\forall j \geq 0$

(B4) $u(x) := x - \tau_0(x)$ is positive, convex and vanishing, as $x \to +\infty$. Also, u'' is decreasing (hence vanishing)

(B5) T preserves the Lebesgue measure m

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(B5) T preserves the Lebesgue measure m

Theorem

Under (B1)–(B5) T is conservative and exact

Types (A) and (B) are of the same nature:

Given $T_o : (0,1) \longrightarrow (0,1)$ preserving μ with $\mu((0,1)) = \infty$, set $\Phi(x) := \mu([x,1])$, for $0 < x < 1$.

By construction $\Phi : (0,1) \longrightarrow \mathbb{R}^+$ pushes μ to the Lebesgue measure m on $\mathbb{R}^+$. Hence $T := \Phi \circ T_o \circ \Phi^{-1} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ has an indifferent fixed point at $+\infty$ and preserves m

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But the two classes are not the same:

Conjugation might not preserve smoothness or expansivity

Important example belonging to both classes: Farey map

$$T_o(x) = \begin{cases} \dfrac{x}{1-x}, & \text{for } x \in \left[0, \frac{1}{2}\right]; \\ \dfrac{1-x}{x}, & \text{for } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

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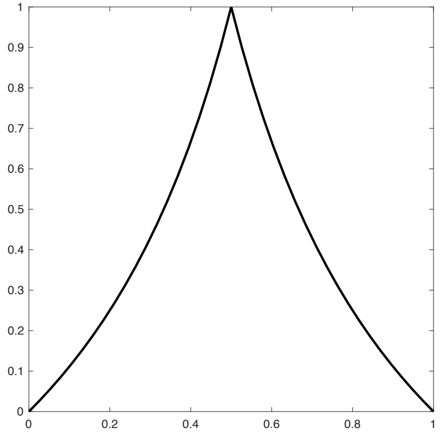
Invariant density: $h(x) = \dfrac{1}{x} \implies \Phi(x) := \displaystyle\int_x^1 h(\xi)\, d\xi = -\log x$

$$T(x) := -\ln(F(e^{-x})) = |\ln(e^x - 1)|$$

Comparison



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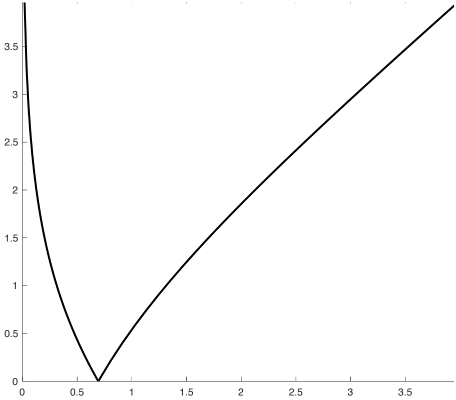

Global observables

Interested in the mixing/stochastic properties of global observables

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Definition (case (A))

$F : (0,1) \longrightarrow \mathbb{C}$ is called global observable if $F \in L^\infty((0,1),\mu)$ and

$$\exists \overline{\mu}(F) := \lim_{a \to 0^+} \frac{1}{\mu([a,1))} \int_a^1 F \, d\mu,$$

Definition (case (B))

$F : \mathbb{R}^+ \longrightarrow \mathbb{C}$ is called global observable if $F \in L^\infty(\mathbb{R}^+, m)$ and

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$\overline{\mu}(F)$ or $\overline{m}(F)$ called infinite-volume average

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Global and local observables

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From now on we give definitions and general facts for case (A) only; case (B) analogous: $((0,1),\mu) \rightsquigarrow (\mathbb{R}^+, m))$ and $\overline{\mu} \rightsquigarrow \overline{m}$

Definition

A local observable is any complex-valued function $f \in L^1$

Global-local mixing

Definition **(GLM2)**

T is global-local mixing if for all global observables F and local observables g

$$\lim_{n\to\infty} \mu((F \circ T^n)g) = \overline{\mu}(F)\mu(g)$$

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In terms of the evolution of measures:

Equivalent definition **(GLM2)**

T is global-local mixing if for all global observables F and probability measures $\nu \ll \mu$

$$\lim_{n\to\infty} T_*^n \nu(F) = \overline{\mu}(F)$$

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In any event,

Proposition

If F is a global observable, so is $F \circ T$, with $\overline{\mu}(F \circ T) = \overline{\mu}(F)$

Global-local mixing, case (A)

Theorem

Let $T : (0,1) \longrightarrow (0,1)$ satisfy (A1)-(A4) with two branches τ_j, $(j = 0, 1)$. Set $\phi_j := (\tau_j)^{-1}$, $h := \frac{d\mu}{dm}$ and assume in addition:

(A5) ϕ_1 decreasing (i.e., τ_1 is decreasing);

(A6) $\phi_0 + \phi_1$ increasing and concave;

(A7) $\phi_0' \, (h \circ \phi_0)/h$ differentiable, strictly decreasing and convex;

(A8) $\phi_0' \, (h \circ \phi_0) + \phi_1' \, (h \circ \phi_1) \geq 0$.

Then T is global-local mixing.

Remark

If h is decreasing, (A8) follows from (A6)

Examples, case (A)

Examples: Farey and friends. For $0 < \alpha < 1$ (also $\alpha = 0$)

$$\phi_0(x) := \frac{x}{(1+x)^{1-\alpha}} \quad ; \quad \phi_1(x) := \frac{1}{(1+x)^{1-\alpha}}$$

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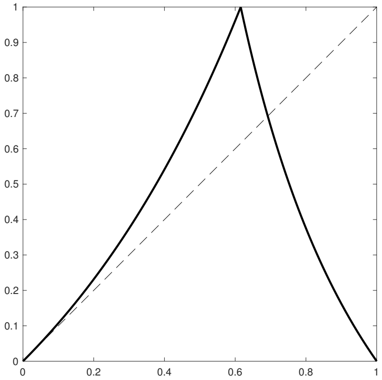
$\alpha = 0.3$



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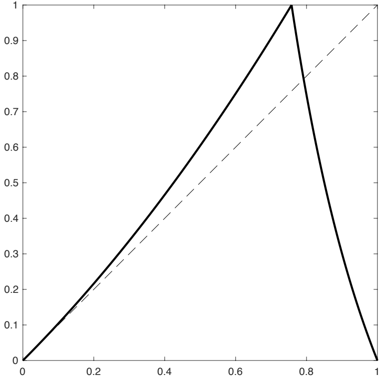
$\alpha = 0.6$



Remark

Theorem generalizes to $N-1$ increasing convex $+$ 1 decreasing branches with similar assumptions

Global-local mixing, case (B)

Theorem

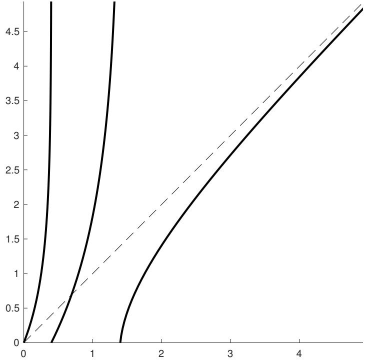
Let $T : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfy (B1)-(B5) (no limit on number of branches) and assume in addition

(B6) τ_j is increasing and convex $\forall j \geq 1$.

Then T is global-local mixing.

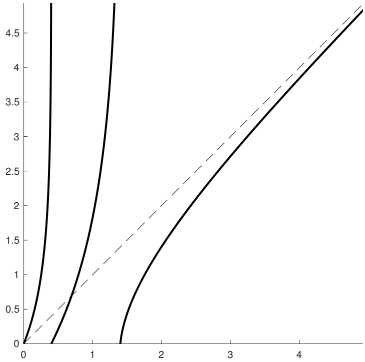
Global-local mixing, case (B)

Example:



Global-local mixing, case (B)

Example:



Remark

Generalizes to 1 increasing and 1 decreasing full branches, with cumbersome assumptions

Applications

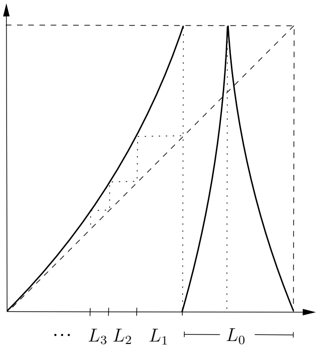
Definition

Given

- $(\mathcal{M}, \mu)$ σ-finite measure space
- $F_n : \mathcal{M} \longrightarrow \mathbb{R}$ measurable $\forall n$
- X random variable on some probability space

one says that $F_n \to X$ strongly in distribution, as $n \to \infty$, if $\forall \nu \ll \mu$, the distribution of F_n w.r.t. ν converges to that of X.

Take T global-local mixing of type (A) [or (B)]



$F_q(x) :=$ hitting time of x to L_0 mod $q \in \mathbb{Z}^+$, i.e.,

$$F_q|_{L_k} \equiv j \quad \Longleftrightarrow \quad k \cong j \,(\mathrm{mod}\ q).$$

Equidistribution of hitting times in residue classes

Proposition

$F_q \circ T^n$ converges strongly in distribution to the uniform random variable on $\{0, 1, \ldots, q-1\}$

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Proof. By global-local mixing,

$$\lim_{n\to\infty} \nu(e^{i\theta F_q \circ T^n}) = \lim_{n\to\infty} T_*^n \nu(e^{i\theta F_q}) = \overline{\mu}(e^{i\theta F_q}) = \frac{1}{q} \sum_{j=0}^{q-1} e^{i\theta j},$$

which is the characteristic function of the uniform variable on $\{0, 1, \ldots, q-1\}$ (last equality is a simple fact). **Q.E.D.**

Partial Birkhoff averaging does not tighten variables

On $((0,1),\mu)$ define the distance $d_\mu(x,y) := \mu([x,y])$

Proposition

Let T be a global-local mixing map of type (A) [or (B)] and F a real-valued global observable s.t.

- F d_μ-uniformly continuous w.r.t. μ [or uniformly continuous]
- $\overline{\mu}(e^{i\theta F})$ [or $\overline{m}(e^{i\theta F})$] exists for all $\theta \in \mathbb{R}$

Then:

1. As $n \to \infty$, $F \circ T^n$ converges strongly in distribution to the variable X with characteristic function $\theta \mapsto \overline{\mu}(e^{i\theta F})$
2. Fix $k \in \mathbb{Z}^+$, $\frac{1}{k}\mathcal{S}_k F \circ T^n \to X$ strongly in distribution
3. $\exists (k_n) \subset \mathbb{Z}^+$, $k_n \nearrow \infty$, s.t. $\frac{1}{k_n}\mathcal{S}_{k_n} F \circ T^n \to X$ strongly in distribution,

Partial Birkhoff averaging does not tighten variables

Cannot happen for probability-preserving mixing systems!

In fact, given any probability-preserving mixing dynamical system $(\mathcal{M}, \mu, T)$, let f be a non-constant bounded (hence local) observable and call X the random variable defined by f w.r.t. μ:

1. As $n \to \infty$, $\frac{1}{k}\mathcal{S}_k f \circ T^n$ converges strongly in distribution to a variable that, for large k, has a smaller variance than X
2. For any increasing sequence $(k_n) \subset \mathbb{Z}^+$, $\frac{1}{k_n}\mathcal{S}_{k_n} f \circ T^n$ does not converge strongly in distribution to X
3. $\exists$ increasing sequence (k_n), s.t. $\frac{1}{k_n}\mathcal{S}_{k_n} f \circ T^n \to \mu(f) =$ const., strongly in distribution

Partial Birkhoff averaging does not tighten variables

Let us show, e.g..

1. As $n \to \infty$, $\frac{1}{k}\mathcal{S}_k f \circ T^n \to X$ converges strongly in distribution to a variable that, for large k, has a smaller variance than X

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Let us show, e.g..

1. As $n \to \infty$, $\frac{1}{k}\mathcal{S}_k f \circ T^n \to X$ converges strongly in distribution to a variable that, for large k, has a smaller variance than X

Take probability $\nu \ll \mu$. By mixing, for all Borel sets A

$$\nu\left(1_A \circ \tfrac{1}{k}\mathcal{S}_k f \circ T^n\right) = \mu\left(\left(1_A \circ \tfrac{1}{k}\mathcal{S}_k f \circ T^n\right) \tfrac{d\nu}{d\mu}\right) \to \mu\left(1_A \circ \tfrac{1}{k}\mathcal{S}_k f\right)$$

I.e., $\mathrm{distr}_\nu\left(\frac{1}{k}\mathcal{S}_k f \circ T^n\right) \to \mathrm{distr}_\mu\left(\frac{1}{k}\mathcal{S}_k f\right)$. Again by mixing, for all sufficiently large j,

$$\left|\mu\left([f \circ T^j - \mu(f)]\,[f - \mu(f)]\right)\right| < \mu\left([f - \mu(f)]^2\right) > 0$$

whence, for k large enough,

$$\mu\left(\left[\tfrac{1}{k}\mathcal{S}_k f - \mu(f)\right]^2\right) < \mu\left([f - \mu(f)]^2\right)$$

Q.E.D.

Thank you!

Thank you!

Happy Birthday, Lyonia!