Thermal regularization in fluid equations.

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  - Background and Navier-Stokes-Fourier System
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$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0$$
$$\partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x p = \mathrm{div}_x \mathbb{S}$$
$$\partial_t(\rho e) + \mathrm{div}_x(\rho e u) + \mathrm{div}_x q = \mathbb{S} : \nabla_x u - p\,\mathrm{div}_x u$$

where p is the pressure, q is the heat flux, $\mathbb{S}$ is the viscous stress tensor, and e is the internal energy.

Navier-Stokes-Fourier system

The viscous stress tensor is given by Newton's law:

$$\mathbb{S}(\theta, \nabla_x u) = \mu(\rho, \theta)\left(\nabla_x u + (\nabla_x u)^T - \frac{2}{3} I \mathrm{div}_x u\right) + \eta(\rho, \theta) I \mathrm{div}_x u.$$

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See the lecture notes of [Novotny, 2012].

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Recent work also focuses on ‘’dissipative measure-valued solutions” for the NSF system, and their stability properties; see [Brezina-Feireisl-Novotny, 2018].

Thermally enhanced dissipation

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Short-circuited model [Ladyzhenskaya, 1970] [Du-Gunzburger, 1991]:

$$\partial_t u + u \cdot \nabla u + \nabla p - \nabla \cdot (A(u)\nabla u) = f$$

$$A(u) = \nu_0 + \nu_1 |\nabla u|^r, \quad r > 0.$$

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This equation is globally well-posed since it satisfies a maximum principle; see [Unterberger 2015]. A similar bound would hold for the "thermal" version of the above equation. To work in a context where we have no better initial energy estimate, we introduce the reduced Burgers' equation

$$\partial_t u + u \cdot \nabla u + \frac{1}{2}(\nabla \cdot u)u - \nu \Delta u = 0.$$

Simplified thermal fluid equations

From this we build the thermal reduced Burgers' equation:

$$\partial_t u + u \cdot \nabla u + \frac{1}{2}\mathrm{div}(u)u - \nu \mathrm{div}(\theta \nabla u) = 0 \tag{1}$$

$$\partial_t(\theta^2) + \mathrm{div}(u\theta^2) - \kappa \mathrm{div}(\theta \nabla(\theta^2)) = \nu\theta|\nabla u|^2 \tag{2}$$

on the domain $\mathbb{T}^3$ with initial data u_0 and $\theta_0 \geq 1$.

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Note the similarities with the NSF system. Here θ^2 represents the "heat energy density", which is roughly like ρe (the density and specific heat capacity are constant). Also, $\nu\theta\nabla u$ models the viscous stress tensor $\mathbb{S}$.

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The choice of thermal viscosity is motivated by an empirical formula for gases [Lautrup, 2011].

Simplified thermal fluid equations

As long as the heat-density remains positive, we can write (2) as

$$\partial_t \theta + u \cdot \nabla \theta + \frac{1}{2} \theta \mathrm{div} u - \kappa \theta \Delta \theta - 2\kappa |\nabla \theta|^2 = \frac{\nu}{2} |\nabla u|^2. \quad (3)$$

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Let $K = \kappa/\nu + 1$ be (a shift of) the inverse Prandtl number. This is dimensionless and will frequently appear in the calculations that follow.

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In order to justify (3), we need to know θ^2 stays positive. But compressibility allows for an expanding gas to become colder (refrigeration).

Initial thermodynamic estimates

Assume that u and θ^2 are classical solutions of the model (1)-(2) on $\mathbb{T}^3 \times [0, T]$ with initial data u_0 and $\theta_0 \geq 1$.

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Lemma (T 3.1)

For u and θ^2 as above, we have $\inf_{\mathbb{T}^3} \theta^2(\cdot, t) \geq 1/(3t/(8\nu) + 1)^2$ and $E_t = E_0$ for all $t \in [0, T]$.

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Intuitively, if $\theta(\cdot, t)$ has a minimum at $\bar{x}$, then (3) implies

$$\partial_t \theta(\bar{x}, t) \geq \frac{\nu}{2}|\nabla u|^2 - \frac{1}{2}|\mathrm{div}(u)|\theta(\bar{x}, t) \geq -\frac{3}{8\nu}\theta(\bar{x}, t)^2.$$

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Rigorously, we prove a minimum principle for $v(x, t) := (1 + 3t/(8\nu))\theta(x, t)$.

Thermally weighted enstrophy estimates

The main result is the following improved a priori estimate.

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Theorem (T 1.2)

Assume that $K \geq 2$. *For* u *and* θ *as above with* $u_0 \in H^1(\mathbb{T}^3)$ *and* $\theta_0 \in L^2(\mathbb{T}^3)$, *there exist constants* $C = C(\nu, K, E_0) > 0$ *and* $M = M(K) > 0$ *such that,*

$$\int \theta^{-1/K} |\nabla u|^2 dx \leq \int \theta_0^{-1/K} |\nabla u_0|^2 dx + C(t^M + 1),$$

for all $t \in [0, T]$. *Moreover, the quantities*

$$\theta^{(K-1)/K} |\nabla^2 u|^2, \; \theta^{-(K+1)/K} |\nabla u|^4, \; \theta^{-(K+1)/K} |\nabla u|^2 |\nabla \theta|^2$$

are $L^1_{[0,T]} L^1_{\mathbb{T}^3}$ *(with bounds depending on* E_0, ν, *and* K*).*

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$$\underbrace{\int f \frac{\partial_t}{2} |\nabla u|^2 dx}_{I_0} \underbrace{- \int u_i \partial_i u_j \partial_k (f \partial_k u_j) dx - \frac{1}{2} \int u \text{div}(u) \partial_k (f \partial_k u_j) dx}_{I_A}$$
$$\underbrace{+\nu \int \partial_i (\theta \partial_i u_j) \partial_k (f \partial_k u_j) dx}_{I_D} = 0. \quad (4)$$

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$$\underbrace{+\nu \int \partial_i(\theta\partial_i u_j)\partial_k(f\partial_k u_j)dx}_{I_D} = 0. \qquad (4)$$

Note that I_0 is not a time derivative of a weighted norm.

Thermally weighted enstrophy estimates

The advection term becomes

$$I_A \geq -\int f|\nabla u|^3 dx - \frac{1}{2}\int f'|\nabla u|^2 u_i \partial_i \theta dx$$
$$- \frac{1}{2}\int f|\nabla^2 u||\nabla u||u| dx.$$

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The viscosity term becomes

$$I_D = \nu \int (\partial_k \theta \partial_i u_j + \theta \partial_{ki}^2 u_j)(f' \partial_i \theta \partial_k u_j + f \partial_{ik}^2 u_j) dx$$
$$= \underbrace{\nu \int f' |\nabla \theta \cdot \nabla u|^2 dx}_{J_1} + J_2 + \underbrace{\nu \int \theta f |\nabla^2 u|^2 dx}_{J_3}$$

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$$= \underbrace{\nu \int f' |\nabla \theta \cdot \nabla u|^2 dx}_{J_1} + J_2 + \underbrace{\nu \int \theta f |\nabla^2 u|^2 dx}_{J_3}$$

with

$$J_2 = \nu \int (f + \theta f') \partial_k \theta \partial_i u_j \partial^2_{ik} u_j dx.$$

Thermally weighted enstrophy estimates

One final integration by parts gives

$$J_2 = \frac{\nu}{2\kappa} \int \frac{f + \theta f'}{\theta} (-\kappa\theta\Delta\theta) |\nabla u|^2 dx - K_0$$
$$= \frac{1}{2} \int \mathcal{F} \left(\frac{\nu}{2} |\nabla u|^2 + 2\kappa |\nabla\theta|^2 - u \cdot \nabla\theta - \partial_t \theta \right) |\nabla u|^2 dx - K_0$$

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with

$$K_0 = \frac{\nu}{2}\int (2f' + \theta f'')|\nabla\theta|^2|\nabla u|^2 dx$$

and

$$\mathcal{F} = \frac{\nu}{\kappa}\frac{f+\theta f'}{\theta}.$$

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The improvement in the enstrophy estimate is then seen in J_2. It produces a positive (weighted) term with the gradient of u appearing to the fourth power. This is ultimately what allows the estimate to close in a novel way.

Initial thermodynamic estimates (and gap)

Putting it all together, (4) becomes

$$\frac{\partial_t}{2}\int K\theta^{-1/K}|\nabla u|^2dx+\underbrace{\nu K\int \theta^{(K-1)/K}|\nabla^2 u|^2dx}_{U_1}$$

$$+\quad \underbrace{\nu\frac{2K^2-3K-1}{2K}\int \theta^{-(K+1)/K}|\nabla\theta|^2|\nabla u|^2dx}_{U_2}$$

$$+\quad \underbrace{\frac{\nu}{4}\int \theta^{-(K+1)/K}|\nabla u|^4dx}_{U_3}$$

$$\le \underbrace{CK\int \theta^{-1/K}|\nabla u|^3dx}_{R_1}+\underbrace{CK\int \theta^{-1/K}|u||\nabla u||\nabla^2 u|dx}_{R_2}\,.$$

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$$R_1 \lesssim \left(\int |\nabla u|^6 dx\right)^{\frac{1}{9}} \left(\int \theta^{-(K+1)/K} |\nabla u|^4 dx\right)^{\frac{7}{12}} \left(\int \theta^2 dx\right)^{\frac{7K-5}{12K}}$$

$$\lesssim \left(\int |\nabla^2 u|^2 dx\right)^{\frac{1}{3}} \left(\int \theta^{-(K+1)/K} |\nabla u|^4 dx\right)^{\frac{7}{12}} E(0)^{\frac{7K-5}{12K}}$$

$$\leq \frac{1}{4} U_1 + \frac{1}{4} U_3 + C \frac{K^8}{\nu^{11}} (1 + T^M) E_0^{\frac{7K-5}{K}}$$

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The last term, by Holder's and Young's inequalities, is bounded as

$$R_2 \leq \frac{1}{4}U_1 + \frac{1}{3}U_2 + C\frac{K}{\nu^3}\left(1 + T^M\right)\int |u|^4 dx.$$

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We then use Agmon's inequality on $\mathbb{T}^3$ to write

$$\|u\|_{L^4}^4 \leq \|u\|_{L^2}^{5/2}\left(\int \theta^{\frac{K-1}{K}}|\nabla^2 u|^2 dx + (1 + T^M)\|u\|_{L^2}^2\right)^{3/4}$$

so that

$$C\frac{K}{\nu^3}(1 + T^M)\|u\|_{L^4}^4 \leq \frac{1}{4}U_1 + C\frac{K}{\nu^{15}}(1 + T^M)E_0^5.$$

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We then have that (4) becomes

$$\frac{\partial_t}{2}\int K\theta^{-1/K}|\nabla u|^2 dx + \frac{1}{4}(U_1 + U_2 + U_3) \\ \leq C(\nu, K, \|u_0\|_{L^2}, \|\theta_0\|_{L^2})(1 + T^{M(K)}).$$

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Integrating then gives the main result.

Observe that the bound holds for all K sufficiently large. Hence ν can be arbitrarily small and κ can be arbitrarily large (both counterintuitive to regularity); the constants degenerate in those limits, though.

The incompressible version

Instead of (1)-(2), it is possible to thermalize the Navier-Stokes equations directly, yielding

$$\partial_t u + u \cdot \nabla u + \nabla p - \nu \mathrm{div}(\theta \nabla u) = 0, \quad (5)$$

$$\partial_t(\theta^2) + u \cdot \nabla(\theta^2) - \kappa \mathrm{div}(\theta \nabla(\theta^2)) = \nu\theta|\nabla u|^2 \quad (6)$$

$$\mathrm{div}(u) = 0 \quad (7)$$

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However, the divergence-free condition leads to an exotic pressure term:

$$p = (-\Delta)^{-1}\partial_i(u_j\partial_j u_i - \nu\partial_j\theta\partial_i u_j). \tag{8}$$

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$$\mathrm{div}(u) = 0 \tag{7}$$

However, the divergence-free condition leads to an exotic pressure term:

$$p = (-\Delta)^{-1}\partial_i(u_j \partial_j u_i - \nu \partial_j \theta \partial_i u_j). \tag{8}$$

The initial thermodynamic estimate is much simpler. The minimum of θ^2 is nondecreasing in time.

The incompressible version

If u and θ are classical solutions of (5)-(7) on $\mathbb{T}^3 \times [0, T]$, we get a conditional improved enstrophy bound.

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Theorem (T 1.1)

For u and θ as above with $u_0 \in H^1(\mathbb{T}^3)$ and $\theta_0 \in L^2(\mathbb{T}^3)$ and θ a Muckenhoupt weight uniformly in t, there exists a constant $K_0 > 0$ such that for all $K \geq K_0$ and all $t \in [0, T]$*

$$\int \theta^{-1/K} |\nabla u|^2 dx \leq \int \theta_0^{-1/K} |\nabla u_0|^2 dx + t \frac{C(K)}{\nu^{15}} E_0^7.$$

Moreover, the following quantities are in $L^1_{[0,T]} L^1_{\mathbb{T}^3}$:

$$\theta^{(K-1)/K} |\nabla^2 u|^2, \ \theta^{-(K+1)/K} |\nabla u|^4, \ \theta^{-(K+1)/K} |\nabla u|^2 |\nabla \theta|^2.$$

The incompressible version

The argument proceeds in the same way, but the pressure term becomes

$$I_P = -\int R_l R_k [u_m \partial_m u_l - \nu \partial_m \theta \partial_l u_m] f' \partial_j \theta \partial_k u_j dx,$$

where $R_j = \partial_j (-\Delta)^{-1/2}$ is a Riesz operator.

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Assuming $\theta(\cdot, t)^{\frac{K+1}{2K}}$ is a Muckenhoupt weight uniformly,

$$\begin{aligned} I_P &= \int R_k R_l (u_m \partial_m u_l - \nu \partial_m \theta \partial_l u_m) \theta^{-(K+1)/K} \partial_j \theta \partial_k u_j dx \\ &\lesssim \frac{1}{\nu} \int \theta^{-\frac{K+1}{K}} |u|^2 |\nabla u|^2 dx + (M^2 + 1)\nu \int \theta^{-\frac{K+1}{K}} |\nabla \theta|^2 |\nabla u|^2 dx \\ &\le \frac{1}{\nu} \int \theta^{-\frac{K+1}{K}} |u|^2 |\nabla u|^2 dx + \frac{1}{2} U_2. \end{aligned}$$

OUTLINE

- Introduction
  - Background and Navier-Stokes-Fourier System
  - Thermally enhanced dissipation
- A priori bounds and thermal regularization
  - Simplified thermal fluid model
  - Thermally weighted enstrophy estimates
  - A word on the incompressible version
- Existence of solutions
  - Iteration scheme
  - Recovering the a priori estimate
  - Improved bounds for the temperature

True well-posedness (work in progress)

The previous results were all a priori; given a smooth solution on $\mathbb{T}^3 \times [0, T]$, we have an enstrophy inequality that closes independently of higher-regularity properties of solutions.

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Unfortunately, the proofs above used the structure of the equations in very precise ways.

The iteration scheme

For $\epsilon > 0$, we consider the following system of equations:

$$\partial_t u + u^\epsilon \cdot \nabla u + \frac{1}{2} v^\epsilon u - \nu \mathrm{div}(S \nabla u) + \epsilon |u|^{10} u = 0$$

$$\partial_t \theta + u^\epsilon \cdot \nabla \theta + \frac{1}{2} v^\epsilon \theta - \kappa \left(\frac{S}{\theta} + S' \right) |\nabla \theta|^2 = \frac{\nu S}{2\theta} |\nabla u|^2$$

$$u^\epsilon := \rho_\epsilon * u, \quad v^\epsilon := \mathrm{div}(u^\epsilon)$$

$$S = S_{\gamma,\epsilon}(\theta) := \gamma + \frac{\theta}{1 + \epsilon^8 \theta}.$$

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There are four regularizations taking place here. We will take a limit as $\epsilon \to 0$.

Recovering the a priori estimate

Following the same strategy (with $f = KS^{-1/K}$), we get an analog of (4):

$$\partial_t \frac{K}{2} \int S^{-\frac{1}{K}} |\nabla u|^2 dx + \nu K \int S^{\frac{K-1}{K}} |\nabla^2 u|^2 dx$$

$$+ \frac{\kappa}{2} \int S^{-\frac{K+1}{K}} \frac{|\nabla\theta|^2 |\nabla u|^2}{(1+\epsilon^8\theta)^3} dx + \frac{\nu}{4} \int S^{-\frac{K+1}{K}} \frac{|\nabla u|^4}{(1+\epsilon^8\theta)^3} dx$$

$$\leq 2K \int S^{-\frac{1}{K}} \left(|\nabla u|^2 |\nabla u^\epsilon| + |u||\nabla u^\epsilon||\nabla^2 u| \right) dx$$

$$+ \frac{1}{2} \int S^{-\frac{K+1}{K}} \frac{|u||\nabla u^\epsilon||\nabla u||\nabla\theta|}{(1+\epsilon^8\theta)^2} dx$$

Recovering the a priori estimate

Sobolev embedings are trickier, since we cannot assume that S is a Muckenhoupt weight. Nevertheless, we essentially have

$$\int S^{\frac{K-1}{K}} |\nabla^2 u|^2 dx \gtrsim \left(\int S^{\frac{3K-3}{K}} |\nabla u|^6 \right)^{\frac{1}{3}}$$

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The $W^{1,4}$-term is much weaker, so we must split the integrals into $\{\theta < \epsilon^{-8}\}$ and $\{\theta \geq \epsilon^{-8}\}$.

Furthermore, all integrals have weights, and we cannot bound $\int S^\beta |\nabla u^\epsilon|^\alpha$ by $\int S^\beta |\nabla u|^\alpha$ in general. Instead, we deal with error terms and use the fact that

$$\|h^\epsilon - h\|_{L^2} \leq C\epsilon \|h\|_{H^1}.$$

Improved bounds for the temperature

In the end, the inequality closes independently of ϵ (though still depending on γ). Mollifying the advection ruined the pointwise lower bound on the temperature, which is why γ must be sent to zero later.

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Using the new bounds, we would like to obtain new estimates on θ. The hope is to get $\theta \in L^\infty$. We can easily get $\theta \in L^\infty_t L^{11}_x$, which is not quite enough.

The right-hand-side of the θ-equation is still not regular enough to complete the bootstrap, indicating that an analogous estimate (i.e., thermally-weighted H^1) must be performed for θ.

Thank You

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