Non uniform relative equilibria for Euler equations

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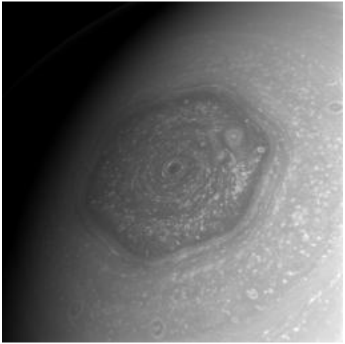
- Saturn's hexagon



Plan

1. Preliminaries on Euler equations.
2. Rotating patches.
3. Non uniform rotating vortices.

2d Euler equations : vorticity formulation

- The vorticity $\omega = \partial_1 v^2 - \partial_2 v^1$ satisfies

$$(E) \begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \quad t \geq 0, x \in \mathbb{R}^2 \\ v = \nabla^\perp \Delta^{-1} \omega \\ \omega_{|t=0} = \omega_0 \end{cases}$$

- Biot-Savart law

$$v(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t,y) dy, \quad x^\perp = e^{i\frac{\pi}{2}} x$$

- Conservation laws :

$$\forall p \in [1, \infty], \forall t \geq 0 \quad \|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}$$

- Classical solutions are global.

Yudovich solutions

- Yudovich (1963) : If $\omega_0 \in L^1 \cap L^\infty$ then (E) has a unique global solution $\omega \in L^\infty(\mathbb{R}_+; L^1 \cap L^\infty)$ and

$$\omega(t,x) = \omega_0(\phi^{-1}(t,x))$$

- A patch is $\omega_0 = \mathbf{1}_D$, with D a bounded domain.

$$\omega(t) = \mathbf{1}_{D_t}, \qquad D_t = \phi(t,D).$$

- Vortex patch problem : what about the regularity of the boundary ?

- Contour dynamics equation (Deem Zabusky 1978) : Let $s \in [0, 2\pi] \mapsto \gamma_t(s)$ be the Lagrangian parametrization is given by : $\partial_t \gamma_t = v(t, \gamma_t)$, then

$$\partial_t \gamma_t(s) = -\frac{1}{2\pi} \int_{\partial D_t} \log \big| \gamma_t(s) - z \big| \, dz.$$

- Regularity persistance : Chemin(1993), Bertozzi-Constantin(1993)

$$\partial D \in C^{1+\varepsilon} \Longrightarrow \forall t \geq 0 \quad \partial D_t \in C^{1+\varepsilon}.$$

- The cases C^1 and Lip are open even locally in time.

Rotating vortice

- Rotating vortice with the angular velocity Ω are solutions in the form :

$$\omega(t,x) = \omega_0\big(e^{-i\Omega t}x\big)$$

- The equation of ω_0 is given by

$$\big(v_0(x) - \Omega x^\perp\big)\cdot\nabla\omega_0(x) = 0,$$

with

$$v_0(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\omega_0(y)dy$$

- Examples :
    - Radial solutions (they rotate with any angular velocity).
    - Kirchhoff ellipses (1876). An elliptic patch rotates uniformly about its centre.

Rotating patche

- A patch $\omega(t) = \mathbf{1}_{D_t}$ is *rotating* with the angular velocity Ω if

$$D_t = e^{i\,\Omega t} D.$$

- The boundary equation is given by

$$\big(v(x) - \Omega x^{\perp}\big) \cdot n(x) = 0, \quad \forall\, x \in \partial D.$$

where n is a normal vector to the boundary. By Green-Stokes theorem

$$\begin{aligned}\overline{v(z)} &= \frac{1}{2i\pi} \int_D \frac{dA(w)}{z - w} \\ &= \frac{1}{4\pi} \int_{\partial D} \frac{\overline{z} - \overline{\xi}}{z - \xi} d\xi\end{aligned}$$

Hence

$$\mathrm{Re}\Big\{\Big(\frac{1}{2i\pi} \int_{\partial D} \frac{\overline{z} - \overline{\xi}}{z - \xi} d\xi + 2\Omega\overline{z}\Big)\vec{\tau}(z)\Big\}, \quad \forall z \in \partial D$$

Potential reformulation

- Recall that the boundary equation is given by the strong formulation

$$(v(x) - \Omega x^{\perp}) \cdot n(x) = 0, \quad \forall\, x \in \partial D.$$

- Note that $v = \nabla^{\perp}\psi$ with ψ the stream function

$$\Delta\psi = \omega = \mathbf{1}_D, \quad \psi(x) = \frac{1}{2\pi}\int_D \log|x - y|dA(y)$$

- Integrating we get the weak formulation

$$\frac{1}{2}\Omega|x|^2 - \frac{1}{2\pi}\int_D \log|x - y|dy - \mu = 0, \quad \forall x \in \partial D.$$

Trivial solutions (simply connected domains)

1. Fraenkel (2000) : let D be a solution with $\Omega = 0$ then D must be a disc.
2. H. (2014) : let D be a convex solution with $\Omega < 0$ then D must be a disc.
3. Let let D be a solution with $\Omega = \frac{1}{2}$ then D must be a disc.

- The proof for $\Omega \leq 0$ is based on the moving plane method.
- The case $\Omega = \frac{1}{2}$ is elementary :
  - the relative stream function $\frac{1}{4}|x|^2 - \psi(x) - \mu$ is harmonic inside D, so it vanishes in D.

$$\forall z \in \mathbb{C}, \partial_z \psi = \frac{1}{4\pi} \int_D \frac{1}{z-y} dA(y) = \left(\frac{1}{4}\overline{z}, \quad \forall z \in D \right)$$

  - By holomorphy we get

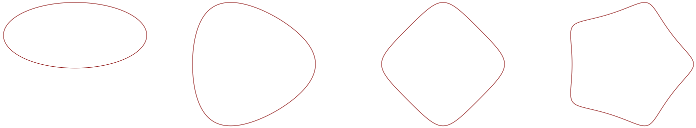
$$z\partial_z \psi = Cte, \forall z \in D^c$$

Nontrivial solutions

1. Kirchhoff vortex (1876). Any *ellipse* with semi-axes a and b rotates with

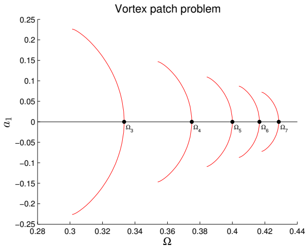
$$\Omega = \frac{ab}{(a+b)2}.$$

2. Numerical observation Deem-Zabusky 1978 : existence of $m-$fold V-states (same symmetry of regular polygon with m sides).



Burbea's re (1982)

- There exists a family of rotating patches $(V_m)_{m\geq 2}$ bifurcating from the disc at the spectrum $\Omega \in \{\frac{m-1}{2m}, m \geq 2\}$. Each point of V_m describes a V-state with m-fold symmetry .



- The case $m = 2$ corresponds to Kirchhoff ellipses.
- Numerical experiments : for $m \geq 3$ the limiting shapes develop right angles.

Stationary bifurcation in infinite dimension

- Consider two Banach spaces X, Y and
$$F : \mathbb{R} \times X \to Y$$
a smooth function such that
$$F(\Omega, 0) = 0, \quad \forall \Omega \in \mathbb{R}$$
- If $\partial_x F(\Omega, 0) \in \mathrm{Isom}(X, Y)$ then by the implicit function theorem, there is no bifurcation at Ω.
- Bifurcation may occur when 0 is an eigenvalue for $\partial_x F(\Omega, 0))$

Crandall-Rabinowitz theorem

Let X, Y be two Banach spaces and

$$F : \mathbb{R} \times X \to Y$$

be a smooth function such that

1. $F(\Omega, 0) = 0, \quad \forall \Omega \in \mathbb{R}$
2. The operator $\partial_x F(0,0)$ is Fredholm of zero index and Ker $\partial_x F(0,0) = \langle x_0 \rangle$ is one-dimensional.
3. Transversality assumption :

$$\partial_\Omega \partial_x F(0,0) x_0 \notin R(\partial_x F(0,0))$$

Then there is a curve of non trivial solutions $s \in (-\varepsilon, \varepsilon) \mapsto (\Omega(s), x(s))$ with

$$\forall s \in (-\varepsilon, \varepsilon), \quad F(\Omega(s), x(s)) = 0$$

General approach

- The boundary is subject to the equation

$$\mathrm{Re}\left\{\left(2\Omega\,\overline{z}+\frac{1}{2i\pi}\int_{\partial D}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi\right)\vec{\tau}(z)\right\}=0,\quad \forall\, z\in\partial D.$$

- Let $\Phi:\mathbb{T}\to\partial D$ be the *conformal parametrization*

$$\Phi(w)=w+\sum_{n\geq 0}\frac{a_n}{w^n},\quad a_n\in\mathbb{R}.$$

We have assumed that the real axis is an axis of symmetry of D.

- Then for any $w\in\mathbb{T}$

$$\begin{aligned} F(\Omega,\Phi(w)) &\equiv \mathrm{Im}\left\{\left(2\Omega\,\overline{\Phi(w)}+\frac{1}{2i\pi}\int_{\mathbb{T}}\frac{\overline{\Phi(\xi)}-\overline{\Phi(w)}}{\Phi(\xi)-\Phi(w)}\Phi'(\xi)d\xi\right)w\,\Phi'(w)\right\}\\ &= 0. \end{aligned}$$

- Recall that for any $w \in \mathbb{T}$

$$F(\Omega, \Phi(w)) \equiv \operatorname{Im}\left\{\left(2\Omega\,\overline{\Phi(w)} + \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{\overline{\Phi(\xi)} - \overline{\Phi(w)}}{\Phi(\xi) - \Phi(w)}\Phi'(\xi)d\xi\right) w\,\Phi'(w)\right\} = 0.$$

- We look for solutions which are small perturbation of the unit disc :

$$\Phi = \mathrm{Id} + f, \quad f(w) = \sum_{n\geq 0} a_n w^{-n}, \, a_n \in \mathbb{R}$$

Remark : With this choice we remove radial solutions and we get rid of the rotation invariance problem.

- Function spaces :

$$X = \left\{f \in C^{1+\alpha}(\mathbb{T})\right\}, \quad Y = \left\{g(w) = \sum_{n\geq 1} b_n \operatorname{Im}(w^n) \in C^{\alpha}(\mathbb{T}), \, b_n \in \mathbb{R}\right\}$$

- For small r, $F : (-1,1) \times B(0,r) \to Y$ is well-defined and smooth.

Spectral study

1. Straightforward computations yield : for $h(w)=\sum_{n\geq 0} a_n w^{-n}\in X$

$$\begin{aligned}\partial_f F(\Omega,0)h(w) &= \frac{d}{dt}F(\Omega,th(w))_{|t=0}\\ &= \sum_{n\geq 1} n\Big(2\Omega-\frac{n-1}{n}\Big)a_{n-1}\mathrm{Im}(w^n)\end{aligned}$$

2. $\Big\{\Omega,\quad \text{Ker}\ \partial_f F(\Omega,0)\neq 0\Big\}=\Big\{\Omega_m:=\frac{m-1}{2m}, m\geq 1\Big\}$ and

$$\text{Ker}\ \partial_f F(\Omega,0)=\langle v_m\rangle,\quad v_m(w)=\overline{w}^{m-1}$$

3. Transversality condition

$$\begin{aligned}\partial_\Omega\partial_f F(\Omega_m,0)v_m &= 2m\,\mathrm{Im}(w^m)\\ &= \notin R(\partial_f F(\Omega_m,0))\end{aligned}$$

Other perspective

1. Boundary regularity of the rotating patches : H.-Mateu-Verdera [2013], Castro, Córdoba, Gómez-Serrano [2015], Hassainia, Masmoudi, Wheeler[2017].
2. Bifurcation and regularity from the ellipses HMV2015, CCG2015.
3. Rotating patches with different topological structure : doubly connected, pairs of patches, multipoles,..
4. Boundary effects : Disc, half plane,..
5. Extension to different transport equations : gSQG, SWQG,Multi-layers fluid.

Non uniform Rotating vortice

- Rotating vortice with the angular velocity Ω are solutions s.t. :

$$\omega(t,x) = \omega_0\big(e^{-i\Omega t}x\big)$$

- The equation of ω_0 is given by

$$F(\Omega,\omega_0)(x) \triangleq (v_0(x) - \Omega x^{\perp}) \cdot \nabla\omega_0(x) = 0,$$

with

$$v_0(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2}\omega_0(y)dy$$

- Problems :
  - What about the existence of non uniform rotating vortices ?
  - Can we find them around any radial profile ?

Main difficultie

- Take ω_0 be a radial smooth solution
- The linearization of F around this radial profile takes the form

$$\mathcal{L}h = \Big(\frac{v_\theta^0}{r} - \Omega\Big)\partial_\theta h + \mathcal{K}(h) \cdot \nabla\omega_0$$

$$\mathcal{K}(h)(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} h(y)dy$$

There are at least two difficulties with this formulation :

1. $Ker(\mathcal{L})$ contains smooth radial functions.
2. The functional F undergoes one derivative loss in all the directions, but the loss for $\mathcal{L}$ is only in the angular direction !

- Castro, Córdoba, Gómez-Serrano [2017] : existence of C^2 rotating vortices with m-fold symmetry $m \geq 2$. They use the level sets reformulation + desingularization of the vortex patch problem.
- Our goal : We shall look for concentrated rotating vortices :

$$\omega(x) = g(x)\,\mathbf{1}_D, \quad v(x) = \frac{1}{2\pi}\int_D \frac{(x-y)^\perp}{|x-y|^2} g(y)dy$$

with g a smooth non constant profile, close to a given radial profile. The domain D is a perturbation of the unit disc.

Reformulation of the equations

- Assume that g is not vanishing on ∂D then

$$\begin{cases} \big(v - \Omega x^{\perp}\big) \cdot n(x) = 0, \quad x \in \partial D, & (1) : \text{boundary equation} \\ \big(v - \Omega x^{\perp}\big) \cdot \nabla g(x) = 0, \quad x \in D, & (2) : \text{density equation} \end{cases}$$

- Let $\Phi : \mathbb{D} \mapsto D$ be the conformal parametrization and set $f : \mathbb{D} \to \mathbb{R}$

$$f = g \circ \Phi$$

- The first equation becomes : $\forall w \in \mathbb{T}$,

$$F(\Omega, f, \Phi)(w) \equiv \mathrm{Im}\left\{\left(\Omega\,\overline{\Phi(w)} - \frac{1}{2\pi}\int_{\mathbb{D}} \frac{f(y)}{\Phi(w) - \Phi(y)} |\Phi'(y)|^2 dy\right) w\,\Phi'(w)\right\} = 0$$

Density equation

- Let ψ be the stream function defined by

$$\psi(x) = \frac{1}{2\pi}\int_D \log|x-y|\, g(y)dy$$

- The density equation is

$$\nabla^\perp\Big[\psi(x) - \frac{1}{2}\Omega|x|^2\Big]\cdot\nabla g(x) = 0$$

- We look for solutions such that

$$\nabla g(x) = \mu(\Omega, g(x))\nabla\Big[\psi(x) - \frac{1}{2}\Omega|x|^2\Big]$$

  with $\mu : \mathbb{R}^2 \to \mathbb{R}$ to be determined.

- Set

$$\mathcal{M}(\Omega, x) \triangleq \int^x \frac{dt}{\mu(\Omega, t)}$$

  then the density equation becomes

$$-\mathcal{M}\big(\Omega, g(x)\big) + \frac{1}{2\pi}\int_D \log|x-y| g(y)dy - \frac{1}{2}\Omega|x|^2 = 0,\ x \in D$$

- Coming back to the unit disc we find

$$G(\Omega,f,\Phi)(z)\triangleq-\mathcal{M}\big(\Omega,f(z)\big)\quad+\quad\frac{1}{2\pi}\int_{\mathbb{D}}\log|\Phi(z)-\Phi(y)|f(y)|\Phi'(y)|^2dy$$
$$-\quad\frac{1}{2}\Omega|\Phi(z)|^2=0,\quad\forall z\in\mathbb{D}.$$

- To fix $\mathcal{M}$ we impose the condition

$$G(\Omega,f_0,\mathrm{Id})(z)=0,\quad\forall\, z\in\mathbb{D},\forall\,\Omega\in\mathbb{R}.$$

Examples

1. Quadratic profiles : $f_0(z)=A|z|^2+B$,

$$\mathcal{M}\big(\Omega,t\big)=\frac{1}{16A}t^2+\frac{B-4\Omega}{8A}t+\frac{3B^2+A^2+4AB-8\Omega B}{16A}$$

2. Gaussian profiles : $f_0(x)=e^{A|z|^2}$,

$$\mathcal{M}\big(\Omega,t\big)=\frac{1}{4A}\Big[-2\Omega t+\int_1^t\frac{s-1}{\ln s}ds\Big]$$

Finally, rotating vortices problem close to the radial solution $f_0 \, \mathbf{1}_{\mathbb{D}}$ reduces to :

$$\begin{cases} F(\Omega, f, \Phi)(w) = \mathrm{Im}\Big\{\Big(\Omega \, \overline{\Phi(w)} - \frac{1}{2\pi}\int_{\mathbb{D}} \frac{f(y)}{\Phi(w)-\Phi(y)}|\Phi'(y)|^2 dy\Big) w \, \Phi'(w)\Big\} = 0 \\ G(\Omega, f, \Phi)(z) = -\mathcal{M}\big(\Omega, f(z)\big) + \frac{1}{2\pi}\int_{\mathbb{D}} \log|\Phi(z)-\Phi(y)| f(y)|\Phi'(y)|^2 dy \\ \qquad -\frac{1}{2}\Omega|\Phi(z)|^2 = 0 \end{cases}$$

From now on, we focus on the quadratic profile : $f_0(z) = A|z|^2 + B$.
We introduce the singular set

$$\mathcal{S}_{\text{sing}} = \left\{ \frac{A}{4} + \frac{B}{2} - \frac{A(n+1)}{2n(n+2)} - \frac{B}{2n}, \quad n \in \mathbb{N}^\star \cup \{+\infty\} \right\}.$$

Theorem (Garcia,H. Soler 2018)

Let $A > 0, B \in \mathbb{R}$. *Then the following results hold true.*

1. *If* $A + B < 0$, *then there is* $m_0 \in \mathbb{N}$ *such that for any* $m \geq m_0$, *there exists a branch of non radial rotating solutions with* $m-$*fold symmetry, bifurcating from the radial solution.*
2. *If* $B > 0$ *and* $m \in \left[1, \frac{B}{A} + \frac{1}{8}\right]$ *or* $m \in \left[1, \frac{2B}{A} - \frac{9}{2}\right]$ *there exists a branch of non radial rotating solutions with* $m-$*fold symmetry. However, there is no bifurcation for any symmetry* $m \geq \frac{2B}{A} + 2$.
3. *If* $B > 0$ *or* $B \leq -\frac{A}{1+\epsilon}$ *for some* $0,0581 < \epsilon < 1$, *then there exists a branch of non radial* $1-$*fold symmetry rotating solutions.*
4. *If* $-\frac{A}{2} < B < 0$ *and* $\Omega \notin \mathcal{S}_{\text{sing}}$, *then there is no solutions close to the quadratic profile.*

Ideas of the proof

- We first solve the boundary equation $F(\Omega, f, \Phi) = 0$ through the implicit function theorem : if $\Omega \notin \mathcal{S}_{\text{sing}}$ then we can locally parametrize the conformal mapping $\Phi = \mathcal{N}(\Omega, f)$, with $\mathcal{N}$ smooth enough.
- We implement bifurcation argument to the equation

$$H(\Omega, f) \triangleq G(\Omega, f, \mathcal{N}(\Omega, f)) = 0$$

1. Identify the dispersion set given by

$$\mathcal{S}_{\text{disp}} = \Big\{\Omega \,:\, \text{Ker}\, \partial_f H(\Omega, f_0) \neq \{0\}\Big\}.$$

2. Dimension of $\text{Ker}\, \partial_f H(\Omega, f_0)$?
3. Transversality assumption ?
4. Separation between $\mathcal{S}_{\text{disp}}$ and $\mathcal{S}_{\text{sing}}$

Structure of the linearized o

- Set $x = \frac{A}{4(\Omega - \frac{B}{2})}$ then

$$\partial_f H(\Omega, f_0) = \Big[\frac{1}{x} - |z|^2\Big]\text{Id} + \mathcal{K}$$

with $\mathcal{K}$ a compact operator.

1. If $x \in (-\infty, 1)$ then $\partial_f H(\Omega, f_0)$ is a Fredholm operator with zero index.
2. If $x \in (1, \infty)$, $\partial_f H(\Omega, f_0)$ is injective but the range is not not closed !

Despite this bad structure, we prove the local uniqueness for the nonlinear problem.

Linearized ω

- The resolution of the kernel equation leads to a Volterra type integro-differential equation. It may solved through transforming it into an ordinary differential equation of second order with polynomial coefficients. This latter one is related to Gauss hypergeometric function

$$F(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},\quad \forall z\in\mathbb{D}.$$

The Pochhammer symbol $(x)_k$ is defined by

$$(x)_k=\frac{\Gamma(x+k)}{\Gamma(x)}$$

- Assume that $\mathrm{Re}(c)>\mathrm{Re}(b)>0$, then

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 s^{b-1}(1-s)^{c-b-1}(1-zs)^{-a}ds,\quad \forall z\in\mathbb{C}\backslash[1,+\infty).$$

Dispersion set

- We obtain the result :

$$\mathcal{S}_{\text{disp}} = \left\{ \Omega = \frac{B}{2} + \frac{A}{4x};\ \exists n \geq 1, x < 1 \quad \text{s.t.} \quad \zeta_n(x) = 0 \right\},$$

with

$$\zeta_n(x) \triangleq F_n(x) \left[1 - x + \frac{A+2B}{A(n+1)} x \right] + \int_0^1 F_n(\tau x) \tau^n \Big[-1 + 2x\tau \Big] d\tau,$$

and

$$F_n(x) = F(a_n, b_n; n+1; x), \quad a_n = \frac{n - \sqrt{n^2+8}}{2}, \quad b_n = \frac{n + \sqrt{n^2+8}}{2}.$$

- What about existence and the distribution of the real roots of $\{\zeta_n\}_n$?

Partial results for $A+B<0$

1. There exists n_0 s.t. for any $n \geq n_0$, there exists only one real solution $x_n \in (0,1)$ to ζ_n
2. The sequence $n \geq n_0 \mapsto x_n$ is strictly increasing converging to 1.
3. Asymptotic behavior :
$$x_n = 1 - \frac{\kappa}{n} + \frac{c_\kappa}{n^2} + o\left(\frac{1}{n^2}\right),$$
where
$$\kappa = -2\frac{A+B}{A} \quad \text{and} \quad c_\kappa = \kappa^2 - 2 + 2\int_0^{+\infty} \frac{e^{-\kappa\tau}}{(1+\tau)^2} d\tau.$$
4. Notice that for the singular set $\mathcal{S}_{\text{sing}} = \left\{\widehat{x}_n, n \in \mathbb{N}^\star \cup \{+\infty\}\right\}$
$$\widehat{x}_n = 1 - \frac{\kappa}{n} + \frac{\kappa^2-2}{n^2} + O\left(\frac{1}{n^3}\right).$$
Thus, for $m \geq m_0$
$$x_m \neq \widehat{x}_{nm}, \ \forall \ n \geq 1$$

Thank you for your attention !