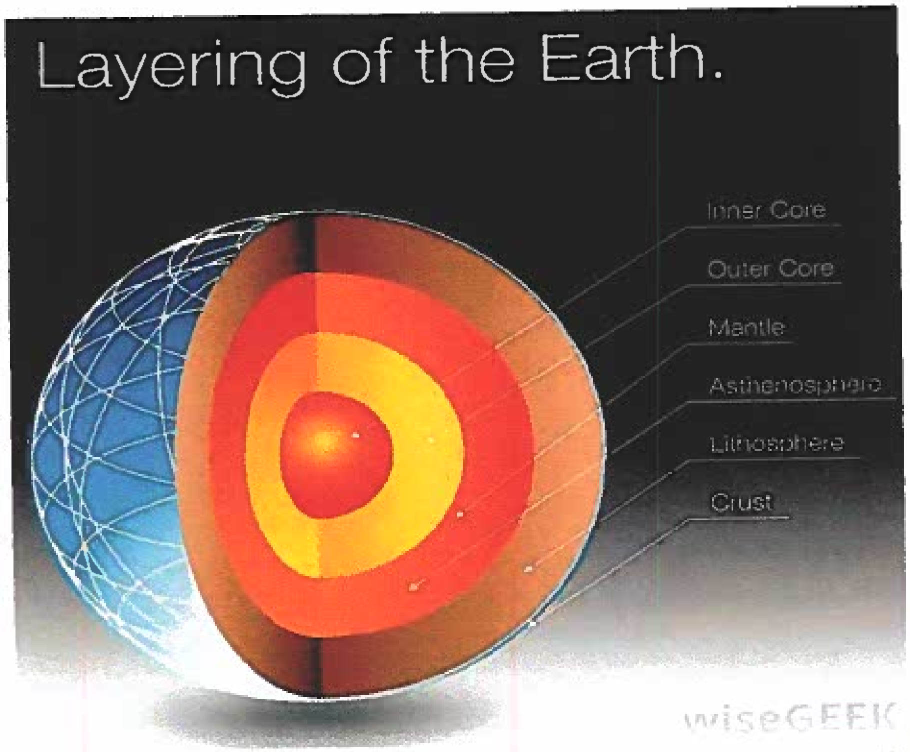
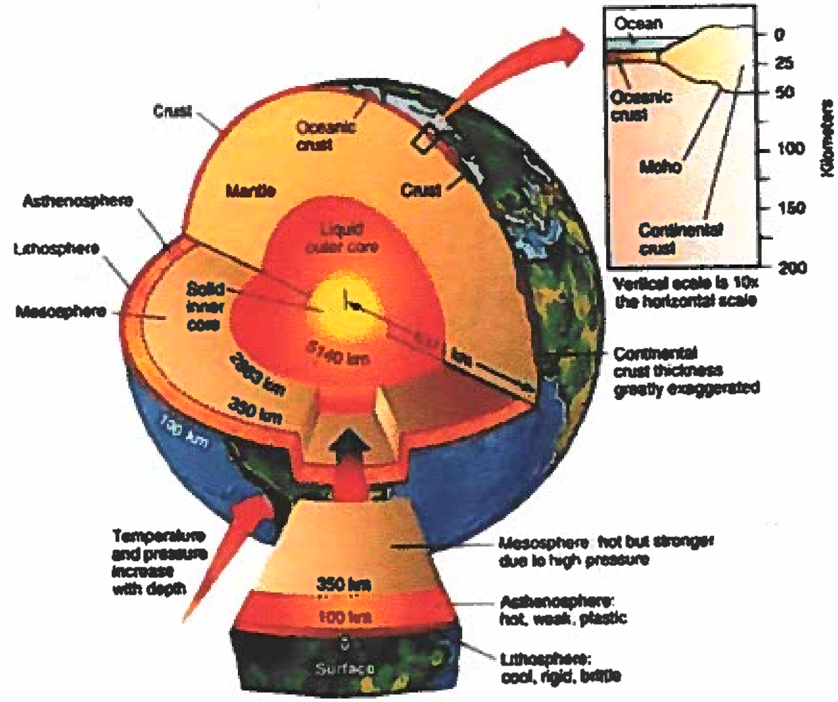
The Joy of Small Parameters

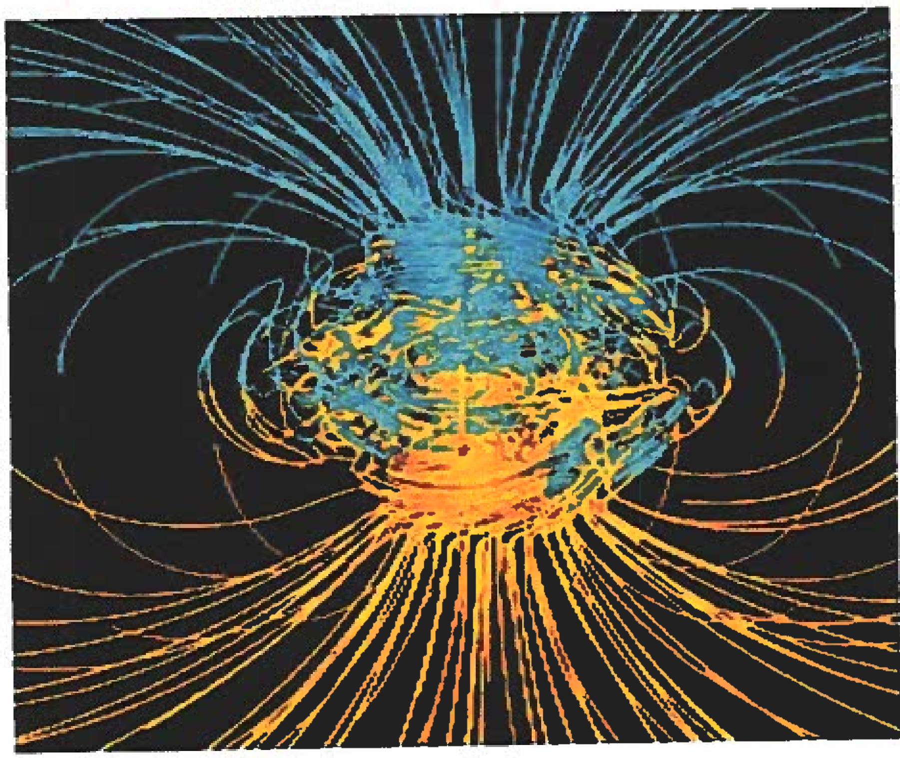
Susan Friedlander, USC
nsf 1613135

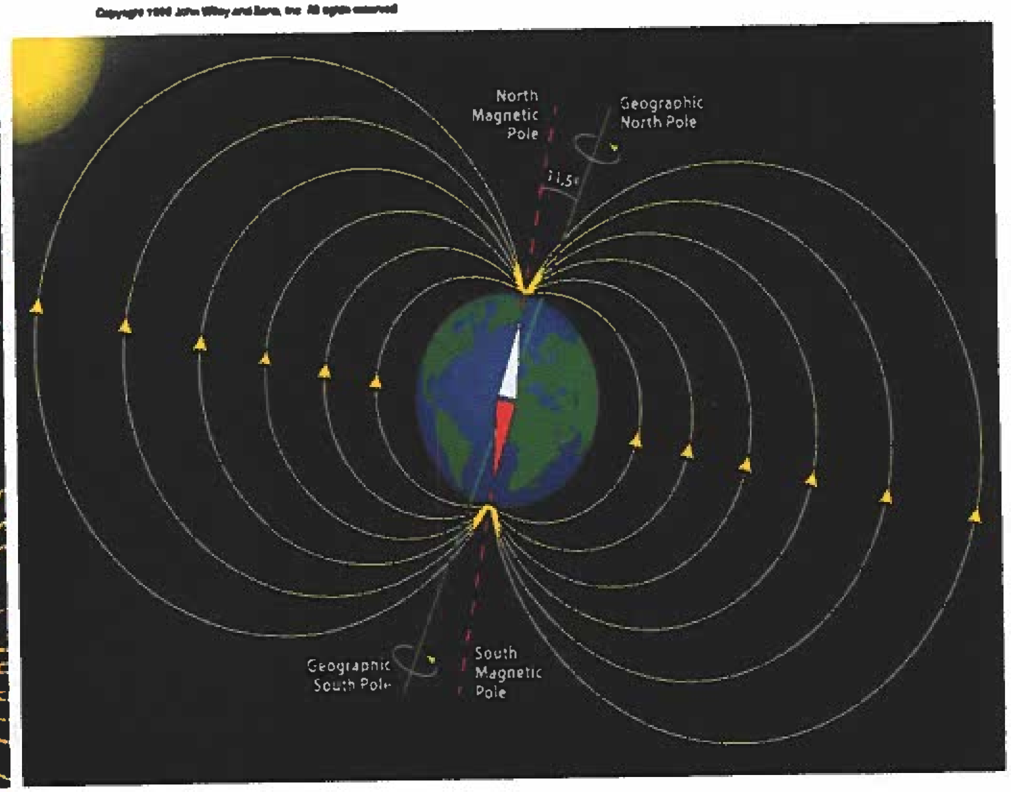
Anthony Suen, HKIE
Hong Kong Early Career 28300016

Nathan Glatt-Holtz, Juraj Foldes, Geordie Richards


Layering of the Earth.
Inner Core
Outer Core
Mantle
Asthenosphere
Lithosphere
Crust
wiseGEEK


Crust
Oceanic crust
Mantle
Crust
Asthenosphere
Lithosphere
Mesosphere
Liquid outer core
Solid inner core
5140 km
2883 km
350 km
100 km
Ocean
Oceanic crust
Moho
Continental crust
0
25
50
100
150
200
Kilometers
Vertical scale is 10x the horizontal scale
Continental crust thickness greatly exaggerated
Temperature and pressure increase with depth
Mesosphere: hot but stronger due to high pressure
350 km
100 km
0
Surface
Asthenosphere: hot, weak, plastic
Lithosphere: cool, rigid, brittle




North Magnetic Pole
Geographic North Pole
Geographic South Pole
South Magnetic Pole

MHD system: Coriolis, Lorentz, Gravity

$$N^2[R_o(\partial_t u + u\cdot\nabla u) + \hat{e}_3 \times u]$$
$$= -\nabla P + \hat{e}_2\cdot\nabla b + R_m\, b\cdot\nabla b + N^2\theta\,\hat{e}_3 + \nu\Delta u$$
$$R_m[\partial_t b + u\cdot\nabla b - b\cdot\nabla u] = \hat{e}_2\cdot\nabla u + \Delta b$$
$$\partial_t\theta + u\cdot\nabla\theta = \kappa\Delta\theta + S$$
$$\nabla\cdot u = 0 \qquad \nabla\cdot b = 0$$

u velocity, b magnetic field, θ temperature

P magnetic + fluid pressure

dimensionless parameters

N^2, R_o, R_m, ν κ

$O(1)$ $O(10^{-3})$ $O(10^{-3})$ very small very small

Moffatt–Loper Model: leading order.

$$N^2 \hat{e}_3 \times u = -\nabla p + \hat{e}_2 \cdot \nabla b + N^2 \theta \hat{e}_3 + \nu \Delta u$$

$$0 = \hat{e}_2 \cdot \nabla u + \Delta b$$

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta + S$$

$$\nabla \cdot u = 0, \quad \nabla \cdot b = 0$$

$$\left\{ \left[\nu \Delta^2 - (\hat{e}_2 \cdot \nabla)^2 \right]^2 + N^4 (\hat{e}_3 \cdot \nabla)^2 \Delta \right\} u$$

$$= -N^2 \left[\nu \Delta^2 - (\hat{e}_2 \cdot \nabla)^2 \right] \nabla \times (\hat{e}_3 \times \nabla \theta)$$

$$+ N^4 (\hat{e}_3 \cdot \nabla) \Delta (e_3 \times \nabla \theta)$$

Domain: $\mathbb{T}^3 \times (0, \infty)$; θ has zero vertical mean

Magnetogeostrophic Equation: MG

$$\partial_t \theta + (u \cdot \nabla)\theta = \kappa \Delta \theta + S$$

$$u = M[\theta]$$

Components of Fourier multiplier symbol $\hat{M}(k)$

$$\hat{M}_1 = [N^4 k_2 k_3 |k|^2 - N^2 k_1 k_3 (k_2^2 + \nu |k|^4)] / D$$

$$\hat{M}_2 = [-N^4 k_1 k_3 |k|^2 - N^2 k_2 k_3 (k_2^2 + \nu |k|^4)] / D$$

$$\hat{M}_3 = [N^2 (k_1^2 + k_2^2)(k_2^2 + \nu |k|^4)] / D$$

where $D = N^4 |k|^2 k_3^2 + (\nu |k|^4 + k_2^2)^2$

recall $k_3 \neq 0$

Observations about the symbol

1) anisotropic, even in wave number k

2) $\underline{\nu > 0}$; $\hat{m}(k) \sim 1/k^2$ as $k \to \infty$

3) $\underline{\nu = 0}$:

On the curved regions in Fourier space $k_3 = O(1)$, $k_1 \sim k_2^2$ and $|k_1| \to \infty$

$$\hat{m}(k) \sim C k_1$$

i.e. when $\nu > 0$ M is smoothing of degree 2

when $\nu = 0$ M is singular of degree -1.

Hierarchy of active scalar equations

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta^\alpha \theta$$

1. Inviscid ($\nu = 0$) MG : singular order 1

$\kappa = 0$, Hadamard ill-posed

$\kappa > 0$, critical case, globally well posed

2. SQG : singular order zero

$\kappa = 0$, open

$\kappa > 0$, critical case, globally well posed

3. 2D Euler in vorticity form: smoothing degree 1

well posed.

4. viscous MG : smoothing degree 2

"better" than 2D Euler

* Inviscid critical MG. ($\nu=0$, $\kappa>0$, $\alpha=1$)

Friedlander & Vicol (2011)

1) Linear parabolic PDE

$$\partial_t \theta + v \cdot \nabla \theta = \kappa \Delta \theta , \quad \nabla \cdot v = 0$$

$$v \in L^2_t L^2_x \cap L^\infty_t BMO^{-1}_x$$

then weak solutions are Hölder continuous

Proof uses De Giorgi iteration.

2) Use this result to prove that Leray-Hopf weak solutions to an active scalar nonlinear PDE of the type * are classical solutions.

Viscous, diffusive MG ($\nu > 0$, $\kappa > 0$)

Th 3. There exists a unique global in time mild solution to

$$\theta_t + u \cdot \nabla \theta = \kappa \Delta \theta, \quad u = M_\nu[\theta], \quad \theta_0 \in L^3$$

such that $\theta \in BC((0,\infty); L^3)$

$$t^{s/2 + 1/2 - 3/2p}\, \theta \in C((0,\infty); \dot{W}^{s,p}), \quad s \in [0,1), \ p \in (3,\infty).$$

In particular, $\theta(\cdot,t) \to \theta_0$ in L^3 as $t \to 0^+$ and $\|\theta(\cdot,t)\|_{\dot{W}^{s,p}} \to 0$ as $t \to \infty$

The solution is instantaneously C^∞ smoothed out and in $\dot{W}^{s,p}$ for all $t > 0$.

The forced MG_ν equation $(\nu \geq 0)$

$(\kappa > 0)$

Friedlander and Suen (2018)

$$\theta_t + u \cdot \nabla \theta = \kappa \Delta \theta + S(x) \qquad (*)$$

$u = M_\nu[\theta]$, $\theta(x,0) = \theta_0(x)$, $S(x) \in C^\infty$ with $\|S\|_{L^\infty} < \infty$

<u>Th 5</u> Let $\theta_0 \in L^2$ and $\kappa > 0$. For all $\nu \geq 0$ there exists a classical solution $\theta_\nu(x,t) \in C^\infty((0,\infty) \times \mathbb{T}^3)$ evolving from θ_0

The proof follows the lines of the proof of global well posedness for the unforced critical MG_0 equation $(\nu = 0)$ given in F.-Vicol (2011)

<u>Th 6</u> (F-Suen, 2018)

Let $\theta_0 \in L^2$, $S \in C^\infty$ and $\kappa > 0$ be given. Then if θ^ν, θ are C^∞ smooth classical solutions of (*) for $\nu > 0$ and $\nu = 0$ respectively with initial data θ_0, then given $\tau > 0$, for all $s \geq 0$, we have

$$\lim_{\nu \to 0} \|(\theta^\nu - \theta)(\cdot, t)\|_{H^s} = 0$$

whenever $t \geq \tau$.

The existence of a global attractor for the forced critical MG_0 equation : $\kappa > 0$, $\nu = 0$

Def^n of a vanishing viscosity weak sol^n.

A weak sol^n of $*$ with $\nu = 0$ is a function $\theta \in C_W([0,T] : L^2(\mathbb{T}^3))$ satisfying $*$ in a distribution:

$$-\int_0^T \langle \theta, \phi_t \rangle dt - \int_0^T \langle u\theta, \nabla\phi \rangle dt + \kappa \int_0^T \langle \nabla\theta, \nabla\phi \rangle dt$$

$$= \langle \theta_0, \phi(x,0) \rangle + \int_0^T \langle S, \phi \rangle dt$$

A weak sol^n to $*$ with $\nu = 0$ is called "vanishing viscosity sol^n" if $\exists$ sequences $\nu_n \to 0$ such that $\{\theta_{\nu_n}\}$ are smooth sol^{n}s and $\theta_{\nu_n} \to \theta$ as $\nu_n \to 0$

<u>Th 7</u>: The system * with $\nu = 0$ possesses a compact global attractor $\mathcal{A}$ in $L^2(\mathbb{T}^3)$:

$\mathcal{A} = \{ \theta_0 : \theta_0 = \theta(0)$ for some bounded complete "vanishing viscosity" solution $\theta(t)$.

For any bounded set $B \subset L^2(\mathbb{T}^3)$ and $\varepsilon, T > 0$ there exists t_0 such that for $t > t_0$ every vanishing viscosity solution $\theta(t)$ with $\theta_0 \in B$ satisfies $\|\theta(t) - x(t)\|_{L^2} < \varepsilon$, $\forall t \in [t_1, t_1 + T]$ for some complete trajectory $x(t) \in \mathcal{A}$, $\forall t \in (-\infty, \infty)$

For $\nu \in [0,1]$ there exists a compact global attractor $\mathcal{A}^\nu$ that is upper semi continuous at $\nu = 0$.

Cheskidov & Dai (2017):

Forced critical SQG possesses a compact global attractor using "classical" viscosity solutions and the abstract framework of evolutionary systems.

For the MG equations asymptotic compactness is not known a priori.

The existence of the global attractor follow from

(i) an energy equality

(ii) the absence of anomalous dissipation for complete bounded trajectories

Földes, F., Glatt-Holtz, Richards (2017)

Stochastically forced "full" MHD system

$\nu > 0$, $\kappa > 0$, $(\lim R_0 \to 0$, $\lim R_m \to 0)$

$$R_0(\partial_t u + u \cdot \nabla u) + \hat{e}_3 \times u =$$
$$-\nabla p + \hat{e}_2 \cdot \nabla B + R_m B \cdot \nabla B + \theta \hat{e}_3 + \nu \Delta u$$
$$R_m(\partial_t B + u \cdot \nabla B - B \cdot \nabla u) = \hat{e}_2 \cdot \nabla u + \Delta B$$
$$d\theta + u \cdot \nabla \theta \, dt = \kappa \Delta \theta \, dt + \sigma \, dW$$

white in time, spatially correlated Gaussian noise $\sigma \, dW$:

$$\sigma \, dW = \sum_{k \in \mathbb{Z}^3} \alpha_k \sigma_k \, dW^k$$

$\alpha_k \in \mathbb{R}$ are the amplitudes

Fundamental postulates of turbulence

Consider energy cascading from large to small spatial scales.

Our system is driven by "spectrally degenerate" stochastic forcing, i.e. noise acts only through a narrow range of low frequencies

Hypo-elliptic situation substantially more difficult than forcing on all spatial scales.

Martingale Solutions for the Full System

(i) Given any initial probability distribution μ there exists a stochastic process $(u, B, \cancel{\theta})$ which is weakly continuous and solves the full evolution system.

(ii) For every $R_o, R_m > 0$ there exists a stationary Martingale solution that satisfies the uniform moment bound

$$\sup_{R_o, R_m \in (0,N]} \mathbb{E} \exp\left(\eta\left(R_o\|u\|^2 + R_m\|B\|^2 + \|\theta\|^2\right)\right) \le C_N < \infty$$

Proved using a Galerkin regularization similar to that used for 3D Navier Stokes

The solutions are weak both in the conventional sense and the probality sense.

The MG$_r$ equation is the limit system obtained with $\nu > 0$, $\kappa > 0$, $R_o = 0$, $R_m = 0$

Very singular limit

Full system supports initial conditions on all variables u, B and θ

The limit system (MG$_r$ active scalar) allows initial conditions only on θ.

Multi-time scale analysis with three time scales:

$O(1)$, $O(R_o^{-1})$, $O(R_m^{-1})$

Results for the limit System.

Stochastically forced MG_r equation, $r>0$, $\kappa>0$.

Well-posedness:

The PDE possesses unique, pathwise Solutions which satisfy exponential Moment bounds.

Extension of known results in the deterministic case

Markovian Dynamics of the limit system.

Use the framework of Hairer-Mattingly

- Show that a form of the Hormander bracket condition is satisfied
- verify a form of asymptotic strong Feller
- an irreducibility condition
- certain exponential moment bounds

We infer that the contractivity property of H-M is satisfied in a suitably chosen Wasserstein metric $\mathcal{W}$.

<u>Th</u>: Let $\{P_t\}_{t\geq 0}$ be the Markov semigroup associated to the MG_r equation.
Then $\{P_t\}_{t\geq 0}$ is contractive in $\mathcal{W}$:

$$\mathcal{W}(\mu_1 P_t, \mu_2 P_t) \leq C e^{-rt} \mathcal{W}(\mu_1, \mu_2), \quad t \geq 0$$

It then follows that $\{P_t\}_{t\geq 0}$ possesses a unique ergodic invariant measure μ.
Furthermore μ satisfies attraction properties:

- it is exponentially mixing
- obeys a strong law of large numbers
- obeys a central limit ~~system~~ theorem

Finite time Convergence as $R_o, R_m \to 0$

We use the powerful observation that if one can establish a contraction property for the limit system in a suitable Wasserstein metric, then the convergence of statistically steady states can be reduced to convergence of solutions of the full system on finite time scales.

Key observation for our problem is to show that a difference in initial conditions on u and B has negligible effect on θ, namely algebraic order in $R_o + R_m$.

Note: convergence for the SSS do not require uniform convergence in u and B up to time $t=0$.

Asymptotics for the full evolution system

Denote variables Θ, U, B for the full system
θ, u, b for the limit system

Results as $R_o \to 0$, $R_m \to 0$

i) Assume $\|\Theta(0) - \theta(0)\| \to 0$

Then for any $t > 0$ $\exists\, \gamma > 0$ and $p > \gamma$ so that

- $$\mathbb{E} \sup_{s \in [0,t]} \|\Theta(s) - \theta(s)\|^p \le C\left(\|\Theta(0) - \theta(0)\| + R_o + R_m\right)^\gamma \to 0$$

  and furthermore

- $$\mathbb{E} \int_0^t \|U(s), B(s) - M_{u,b}[\theta]\|^2 \, ds \le C\left(\|\Theta(0) - \theta(0)\| + R_o + R_m\right)^\gamma \to 0$$

(ii) Convergence of statistically steady states (invariant measures)

- For every $R_o, R_m > 0$ the full system possesses at least one SSS μ_{R_o,R_m} which satisfies certain exponential moment bounds independent of R_o, R_m.
- Any collection μ_{R_o,R_m} converges to μ at an algebraic rate in a suitable metric $\mathcal{W}$.
- In particular for any sufficiently regular observable ψ

$$\left| \int \psi(u, B, \phi)\, d\mu_{R_o,R_m} - \int \psi(M_{u,b}(\theta), \theta)\, d\mu \right| \to 0$$

as $R_o, R_m \to 0$

Thank You for coming!

F.-Vicol, Ann I.H.P. Nonlineaire, 2011

F.-Suen, Nonlinearity, 2015

Foldes, F.-Glatt-Holtz, Richards, SIMA, 2017

F.-Suen, Annals PDE, 2018