Geometric Kinematics and Fluid Interfaces

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- slender jet and shallow water: blow up = pinchoff (with T. Drivas, H. Nguyen, F. Pasqualotto).

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The surface is assumed to be orientable, exterior normal is n. The vectors $\{n, f_{,1}, \cdots, f_{,d}\}$ computed at any $\alpha \in D$ form a basis of $\mathbb{R}^{d+1}$.

Velocity decomposition, evolution of normal

$$v = an + b^j f_{,j}$$

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$$\frac{\partial}{\partial t} f_{,k} = a_{,k} n + a n_{,k} + b^j_{,k} f_{,j} + b^j f_{,jk},$$

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where h_{jk} are the coefficients of the second fundamental form II:

$$h_{jk} = \langle f_{,jk}, n \rangle = - \langle f_{,j}, n_{,k} \rangle$$

Evolution of first and second fundamental forms

Recall $< f_{,pi}, f_{,j} >= [pi, j]$, the Christoffel symbols of the second kind $\Gamma^r_{pi} = g^{rj}[pi, j]$ and

$$b^r_{;i} = b^r_{,i} + \Gamma^r_{pi} b^p,$$

the covariant gradient of the tangent vector b. We obtain after calculations:

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$$\frac{\partial}{\partial t} II = \nabla\nabla a - a\, II(I^{-1})II + L_b(II)$$

where $\nabla\nabla a$ is the matrix:

$$a_{;kl} = a_{,kl} - \Gamma^p_{kl} a_{,p}.$$

and where $L_b(II)$ is the Lie derivative of II given by

$$(L_b(II))_{kl} = b^j h_{kl,j} + b^j_{,k} h_{jl} + b^j_{,l} h_{jk}.$$

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$$\frac{\partial}{\partial t}\sqrt{g} = \left(-adH + \nabla\cdot b\right)\sqrt{g}$$

where the divergence and mean curvature are

$$\nabla\cdot b = b^j_{;j}$$

$$H = \frac{1}{d}\mathrm{Trace}\, W.$$

Note that immersions persist as immersions ($g \neq 0$) as long as the evolution is smooth.

Total area, mean curvature

The total area

$$A = \int \sqrt{g} d\alpha = \int_f dS$$

satisfies

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Using

$$\text{Trace } I^{-1} \nabla\nabla a = \Delta_f(a)$$

and taking the trace of the evolution of the Weingarten map we obtain the equation for H

$$\frac{\partial}{\partial t} H = \frac{1}{d}\Big(a\text{Trace}(W^2) + \Delta_f(a) \Big) + b^j H_{,j}.$$

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where $\widetilde{W} = (\text{Trace } W)\text{Id} - W$. We note

$$\frac{\partial}{\partial t}(K\sqrt{g}) = \sqrt{g}\Big[\text{Trace}\left(\widetilde{W}(I^{-1}\nabla\nabla a + L_b(W))\right) + Kb^j_{;j}\Big]$$

$$= \frac{\partial}{\partial \alpha^i}\left(\sqrt{g}g^{ij}\widetilde{W}^k_j \frac{\partial a}{\partial \alpha^k} + b^i K\sqrt{g}\right)$$

verifies the time independence of the Gauss-Bonnet formula $\int_f KdS = \chi(f)$.

Example: d=1, plane curves.

We write $f(\alpha) = z(\alpha)$. Usual differentiation with respect to the only variable (other than time) is denoted by a prime. The Weingarten matrix is simply the curvature κ of the curve z. The Laplace-Beltrami operator Δ_f is the second derivative with respect to arclength. We obtain:

$$\frac{\partial}{\partial t}\kappa = a\kappa^2 + \frac{d^2}{ds^2}a + b\kappa'$$

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We note that the time invariance of the rotation number $\int_f \kappa ds$ follows: the quantity $q = \kappa|z'|$ obeys the conservation law

$$\frac{\partial}{\partial t}q = \left(|z'|^{-1}a' + bq\right)'.$$

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semilinear heat equation. Self-similar blow up, finite time extinction: $\frac{d}{dt}A = -\int_f \kappa^2 ds$, $\int_f \kappa ds = 1$, Schwartz:

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3) Evolution by arclength derivative of curvature: $a = \kappa_s$, $b = 0$. Length (A) is conserved $\frac{d}{dt}A = 0$. Curvature equation= modified KdV:

$$\partial_t \kappa = \kappa^2 \kappa_s + \frac{d^3}{ds^3}\kappa$$

Does not blow up, completely integrable.

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$$a = n \cdot \nabla p(x,y,t)_{|(x,y)=z(\alpha,t)}$$

is the Dirichlet-to-Neumann of $\gamma\kappa$.

Example, $d = 1$: Irrotational inviscid flow

Irrotational 2d Euler flow. Then $v = \nabla\Phi$. Let Ω be the fluid domain and let $f = \partial\Omega$. Bernoulli:

$$\partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + p = 0$$

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Computing

$$a(\alpha, t) = n \cdot \nabla\Phi(x, y, t)_{|\,(x,y)=z(\alpha,t)}$$

$$b(\alpha, t) = \frac{1}{|z'(\alpha, t)|^2}\partial_\alpha(\Phi(z(\alpha, t), t))$$

The normal derivative $a = \Lambda\phi$, Dirichlet-to-Neumann, $\phi = \Phi_{|\,f}$. If $\gamma = 0$ problem can be ill posed (Ebin). If $\gamma > 0$, pinchoff computed (Day-Hinch-Lister), but problem largely open.

Slender jets

Axisymmetric Navier-Stokes without swirl, with surface tension and gravity. Variables r, x. Interface:

$$r = h(x, t)$$

Boundary conditions:

$$\left(p\mathbb{I} - \nu\left(\nabla v + \nabla v^T\right)\right) \cdot n = \gamma H n$$

Assume: slender jet, i.e. distances across r much smaller than along x. Eggers-Dupont '94: systematic derivation of equations for $h(x, t)$ and axial velocity $u(x, t)$

$$\partial_t h + u\partial_x h = -\frac{1}{2}h\partial_x u,$$
$$\partial_t u + u\partial_x u + \gamma\partial_x(\frac{1}{h}) = 3\nu\frac{\partial_x(h^2\partial_x u)}{h^2} - g,$$

Finite time pinchoff, matching experiments (Nagel et al). Viscous forces cannot be neglected at pinchoff. Irrotationality fails.

Compressible degenerate viscous flow, and active potentials

$$\partial_t \rho + \partial_x(u\rho) = 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2) = -\partial_x p(\rho) + \partial_x(\mu(\rho)\partial_x u) + \rho f$$
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Compressible degenerate viscous flow, and active potentials

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No singularity without pinchoff

Let $\mathbb{T} = [0,1]$. We consider periodic boundary conditions.

Theorem

(Drivas, Nguyen, Pasqualotto, C, '18). Let f be smooth enough,

$$f \in L^2(0, T; H^{k-1}(\mathbb{T})),$$

$k \geq 3$, $T > 0$. *Assume either one of*
A) $c_p > 0$ *and* $\alpha > \frac{1}{2}, \gamma \neq 1, \ \gamma \geq \alpha - \frac{1}{2}$ *(covering viscous shallow water)*

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A) $c_p > 0$ and $\alpha > \frac{1}{2}$, $\gamma \neq 1$, $\gamma \geq \alpha - \frac{1}{2}$ (covering viscous shallow water) or
B) $c_p < 0$ and $\frac{1}{2} < \alpha \leq \frac{3}{2}$, $\gamma < 1$, $0 < \gamma \leq \alpha$ (covering Eggers-Dupont equations).
Then solutions (u,ρ) on $[0,T^*)$ satisfy

$$\sup_{T\in[0,T^*)} \|\rho\|_{L^\infty(0,T;H^k)} + \sup_{T\in[0,T^*)} \|u\|_{L^\infty(0,T;H^k)} + \sup_{T\in[0,T^*)} \|u\|_{L^2(0,T;H^{k+1})} < \infty$$

and can be uniquely continued past T^* if

$$\inf_{t\in[0,T^*)} \min_{x\in\mathbb{T}} \rho(x,t) > 0.$$

Elements of the proof

The proof is technical and uses higher energy metods building on:
Energy

$$e := \frac{1}{2}\rho|u|^2 + \pi(\rho), \qquad \pi(\rho) = \rho \int_{\bar{\rho}}^{\rho} \frac{p(s)}{s^2} ds.$$

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The active potential

$$w = -p(\rho) + \mu(\rho)\partial_x u.$$

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hence the name. The active potential obeys a nonlinear heat equation with nondegenerate or less degenerate diffusivity $\frac{\mu(\rho)}{\rho}$ than the momentum equation. Bounds for the norms of the active potential are obtained using energy estimates, and used to close higher energy estimates for the momentum and density.

Hele-Shaw

Two dimensional potential flow with surface tension. $\Omega \subset \mathbb{R}^2$, $u = \nabla p$, $f = \partial\Omega$, with

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$$\frac{dV}{dt} = \int_{\partial\Omega} \frac{\partial p}{\partial n} dS = 0$$

and

$$\frac{dA}{dt} = -\frac{1}{\gamma}\int_{\partial\Omega} p\frac{\partial p}{\partial n} dS = -\frac{1}{\gamma}\int_{\Omega} |\nabla p|^2 dx < 0$$

Hele-Shaw neck model

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Computations showed self-similar behavior with infinite time pinchoff. Other data lead to finite time pinchoff.

Energy dissipation, steady states

The energy

$$E(h) = \frac{1}{2}\int_I |\partial_x h(x)|^2 dx + P\int_I h(x)dx$$

decays on solutions

$$\frac{d}{dt}E(h(t)) = -D(h(t))$$

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The steady solutions:

$$h_P(x) = \frac{P}{2}(x^2 - 1) + 1,$$

if $P \leq 2$ and

$$h_P(x) = \begin{cases} \frac{P}{2}(|x| - x_P)^2, & \text{for } x_P \leq |x| \leq 1, \\ 0, & \text{for } |x| < x_P \end{cases}$$

for $P > 2$, with $x_P = 1 - \sqrt{\frac{2}{P}}$.

Weak solutions, uniqueness and variational characterization

$$\partial_x(h\partial_x^3 h) = \partial_x^2(h\partial_x^2 h - \frac{1}{2}(\partial_x h)^2).$$

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(CENV '17) The equation has global weak solutions $h(t)$ *which are nonnegative, belong to* C^2 *near the boundary, satisfy the boundary conditions, and are in* $L^2([0, T], H^2(I))$.

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(CENV '17) If $h \geq 0$ and $h \in H^1(I)$ with $h(\pm 1) = 1$ then

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Moreover, $E(h) = E(h_P)$ if and only if $h = h_P$.

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Moreover, $E(h) = E(h_P)$ if and only if $h = h_P$.
Let h_n be a sequence of nonnegative $H^3(I)$ functions satisfying the boundary conditions, which are uniformly bounded in $H^1(I)$ and satisfy $\lim_{n\to\infty} D(h_n) = 0$. Then h_n converge weakly in $H^1(I)$ to h_P and strongly in $H^3_{loc}(\{x \mid\ h_P(x) > 0\})$.

Pinchoff

Theorem

(CENV) **1.** *If $P < 2$ then h_P is asymptotically stable in $H^1(I)$:*

$$\|h(t) - h_P\|_{H^1(I)} \leq C\|h_0 - h_P\|_{H^1(I)} e^{-ct}$$

for $\|h_0 - h_P\|_{H^1(I)} \leq \delta$. Moreover $h(t)$ converge to h_P in $H^3(I)$.

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for $\|h_0 - h_P\|_{H^1(I)} \leq \delta$. Moreover $h(t)$ converge to h_P in $H^3(I)$.
2. *If $P \geq 2$, then starting from positive $h_0 \in H^3(I)$ the solution pinches off in finite time or in infinite time. If the pinchoff is in infinite time then there exists a sequence of times $t_n \to \infty$ such that $h(t_n)$ converges to h_P weakly in $H^1(I)$ and in $H^3_{loc}(\{x \mid \ h_P(x) > 0\})$.*

Local existence, blow up= pinchoff

Let

$$X(T) = L^{\infty}([0,T]; H^3(I)) \cap L^2([0,T]; H^5(I))$$

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(CENV '17) Let $h_0 \in H^3(I)$ be a positive initial datum, satisfying the boundary conditions. Let $m(0) = \min_I h_0(x) > 0$. There exists a positive time $T > 0$ depending only on P, $\|h_0\|_{H^3(I)}$ and $m(0)$ such that the problem has a unique solution $h \in X(T)$ which satisfies $m(T) = \inf_{I\times[0,T]} h(x,t) > 0$.

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$$\|h\|_{X(T)} \leq \mathcal{F}(m(T)^{-1}, \|h_0\|_{H^3(I)})$$

holds with $\mathcal{F}$ a continuous increasing function depending only on P.

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$$\int_0^T D(h(t))dt \leq C(\|h_0\|_{H^3(I)} + 1)$$

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$$\int_0^T D(h(t))dt \leq C(\|h_0\|_{H^3(I)} + 1)$$

so $T = \infty$ triggers convergence to h_P.

Elements of proof, I

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Elements of the proof II

Linear problem

$$\partial_t h + \partial_x (g \partial_x^3 h) = 0$$

with the same boundary conditions ($h(\pm 1, t) = 1$, $\partial_x^2 h(\pm 1, t) = P$).
Take $m_g = \inf_{I \times [0,T]} g(x,t) > 0$.

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$$\|h\|_{X(T)} \leq \mathcal{F}(m_g^{-1}, \|g\|_{L^\infty(I;H^2(I))}, \|\partial_t g\|_{L^1(I;L^\infty(I))}, \|h_0\|_{H^3(I)})$$

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with selfadjoint Neumann-Neumann boundary conditions $\partial_x w(\pm 1, t) = \partial_x^3 w(\pm 1, t) = 0$ which follow from the boundary conditions for $\partial_t h$.

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with selfadjoint Neumann-Neumann boundary conditions $\partial_x w(\pm 1, t) = \partial_x^3 w(\pm 1, t) = 0$ which follow from the boundary conditions for $\partial_t h$. The active potential has therefore very good energy bounds, if $g > 0$ and $\partial_t g$ is no too bad. Approximations, bootstraps, high energy bounds...

Elements of the proof II

Linear problem

$$\partial_t h + \partial_x (g \partial_x^3 h) = 0$$

with the same boundary conditions ($h(\pm 1, t) = 1$, $\partial_x^2 h(\pm 1, t) = P$).
Take $m_g = \inf_{I \times [0,T]} g(x,t) > 0$. Obtain bounds of the form

$$\|h\|_{X(T)} \le \mathcal{F}(m_g^{-1}, \|g\|_{L^\infty(I;H^2(I))}, \|\partial_t g\|_{L^1(I;L^\infty(I))}, \|h_0\|_{H^3(I)})$$

The active potential

$$w = g \partial_x^3 h$$

obeys

$$w_t = -g \partial_x^4 w + \frac{\partial_t g}{g} w$$

with selfadjoint Neumann-Neumann boundary conditions $\partial_x w(\pm 1, t) = \partial_x^3 w(\pm 1, t) = 0$ which follow from the boundary conditions for $\partial_t h$. The active potential has therefore very good energy bounds, if $g > 0$ and $\partial_t g$ is no too bad. Approximations, bootstraps, high energy bounds...

Thank You !