Affine Algebraic Groups, Motives and Cohomological Invariants

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1 Overview of the Field

The theory of affine algebraic groups is a well-established area of modern mathematics. It started as an algebraic version of the massively successful and widely applied theory of Lie groups, pushed forward most notably by Chevalley and Borel. In the hands of Serre and Tits, it developed into a powerful tool for understanding algebra, geometry and number theory (e.g. Galois cohomology). In particular, it provides a way to unify seemingly distinct statements in algebra, geometry and number theory, hence, suggesting new techniques and methods for solving problems in these areas.

For example, the Hasse-Minkowski Theorem and the Albert-Brauer-Hasse-Noether Theorem, which concern respectively quadratic forms and central simple algebras over global fields, can be viewed as special cases of the celebrated Hasse Principle in Galois cohomology of semisimple linear algebraic groups (due to Kneser, Harder, and Chernousov) which unifies these two theorems and provides many new results that would not have been suspected before.

This philosophy has led to a vast number of new techniques that proved useful in many different areas of mathematics such as: quadratic forms (Karpenko, Merkurjev, Vishik), essential and canonical dimensions (Reichstein, Merkurjev), local-global principles (Hartmann, Harbater, Krashen, Parimala, Suresh), motives (Petrov, Semenov, Zainoulline), and torsors (Chernousov, Gille, Panin, Pianzola). For instance, Karpenko proved that some results on isotropy of hermitian forms (related to affine algebraic groups of type A) induce analogous results for symplectic and quadratic forms (groups of types B, C, D), via the theory of algebraic cycles and and the study of Chow motives of projective homogeneous varieties.

Further applications recently emerged in more arithmetic topics, such as the study of arithmetic groups and arithmetic locally symmetric spaces (Prasad, Rapinchuk) and pseudo-reductive groups (Conrad). In the opposite direction, another trend has been using results from finite group theory to prove theorems about algebraic groups (Guralnick).

In this workshop, we brought together specialists and young researchers from these areas, to stimulate new advances and developments, with particular emphasis on recent applications of algebraic groups in algebra, geometry and number theory that establish new links between different areas of mathematics. For example, the following topics have been included: the proof of the Grothendieck-Serre conjecture based on the theory of affine Grassmannians coming from the Langland's program; the breakthrough in the computation of cohomological invariants of degree 3 by Merkurjev based on new results concerning motivic cohomology; Conrad's proof of finiteness of fibers for fppf cohomology over global fields of prime characteristic; and applications of representations with dense orbits inspired by Bhargava's work.

2 Recent Developments and Open Problems

During the last decade, we observed an explosion of research activity in the area of affine algebraic groups and applications. Below, we describe some topics, where striking results were recently obtained.

2.1 The geometric case of the Grothendieck-Serre conjecture (Fedorov-Panin)

Fedorov and Panin have proved this conjecture, stated in the mid-1960's, which says that if a G-torsor (defined over a smooth algebraic variety X) is rationally trivial, then it is locally trivial (in Zariski topology), where G is a smooth reductive group scheme over X. Their proof uses the theory of affine Grassmannians.

This conjecture has a long history: It was first proven for curves and surfaces for quasi-split groups by Nisnevich in the mid-1980's, and for arbitrary tori in the late 1980's by Colliot-Thelene and Sansuc. If G is defined over a field, then the conjecture is known as Serre's conjecture and was proven by Colliot-Thelene, Ojanguren and Raghunathan in the beginning of the 1990's. Ojanguren-Panin-Suslin-Zainoulline proved it for most classical groups in the late 1990's. Finally, Panin-Stavrova-Vavilov gave a proof for isotropic groups. However, no general argument was known up to now. Recently Fedorov and Panin found a new original approach that proves the conjecture in general, using the theory of affine Grassmannians coming from Langlands' program.

2.2 Computation of the group of degree 3 cohomological invariants, using motivic cohomology (Merkurjev)

According to J.-P. Serre, by a cohomological invariant, one means a natural transformation from the first Galois cohomology with coefficients in an algebraic group G (the pointed set which describes all G-torsors) to a cohomology functor $h(-)$, where h is, for instance, some Galois cohomology group with torsion coefficients, a Witt group, or a Chow group with coefficients in a Rost cycle module M. The ideal result here would be to construct enough invariants to classify all G-torsors under some affine algebraic group G. The question was put on a firm foundation by Serre and Rost in the 1990's, allowing the proof of statements like "the collection of cohomological invariants of G is a free module over the following cohomology ring" for certain groups G; this theory is expounded in the 2003 book by Garibaldi-Merkurjev-Serre.

Using this theory one obtains a complete description of all invariants landing in degree 1 Galois cohomology for arbitrary G, in degree 2 if G is connected, and in degree 3 if it is simply connected and semisimple (Rost). In a breakthrough recent development, which is the starting point of important research activity, Merkurjev provided a complete description of degree 3 invariants for semisimple groups, solving a long standing question. The full power of his entirely new methods, based on new results in motivic cohomology, still have to be understood. Moreover, even though he describes the finite group of degree three invariants, we often lack a complete description of generators of this group, that is, new invariants have been discovered, that still have to be described.

2.3 Length spectra of locally symmetric spaces (Prasad and Rapinchuk)

The answer to the question "Can you hear the shape of a drum?" is famously no. Nevertheless, some variations of the problem where one strengthens the hypothesis or restricts the collection of spaces under consideration has led to situations where the answer is yes. In a remarkable 2009 Pub. Math. IHES paper, Prasad and Rapinchuk introduced the notion of weak commensurability of semisimple elements of algebraic groups and of arithmetic groups and used this new concept to address the question of when arithmetically defined locally symmetric spaces have the same length spectrum. In this paper they also settled many cases of the long-standing question of when algebraic groups with the same maximal tori are necessarily isomorphic. This paper, and the stream of research stemming from it, connects algebraic groups and their Galois cohomology – the central subject of this conference – with arithmetic groups, geometry, and even transcendental number theory.

2.4 Applications of algebraic cycles and Grothendieck gamma filtration to invariants of torsors and algebras with involutions (Baek-Garibaldi-Gille-Queguiner-Zainoulline)

Let X be the variety of Borel subgroups of a simple linear algebraic group G over a field k. In a series of papers it was proven that the torsion part of the second quotient of Grothendieck's gamma-filtration on X is closely related to the torsion of the Chow group and hence to the group of cohomological invariants in degree 3 computed recently by Merkurjev. As a byproduct of this new striking connection one obtains an explicit geometric interpretation/description of various cohomological invariants in degree 3 as well as new results concerning algebraic cycles and motives of projective homogeneous spaces.

2.5 Genericity theorems for the essential dimension of algebraic stacks and their applications (Brosnan-Reichstein-Vistoli)

Techniques from the theory of algebraic stacks can be used to prove genericity theorems, that bound the essential dimension of those stacks. As an application, new bounds for the more classical essential dimension problems for algebraic groups, forms and hypersurfaces are obtained. These genericity theorems have also been used in particular by Biswas, Dhillon and Lemire to find bounds on the essential dimension of stacks of (parabolic) vector bundles over curves.

2.6 Some classification results (simple stably Cayley groups, by Borovoi-Kunyavskii-Lemire-Reichstein, and finite groups of low essential dimension, Beauville and Duncan)

A linear algebraic group is called a Cayley group if it is equivariantly birationally isomorphic to its Lie algebra. It is stably Cayley if the product of the group and some torus is Cayley. Cayley gave the first examples of Cayley groups with his Cayley map back in 1846. Over an algebraically closed field of characteristic 0, Cayley and stably Cayley simple groups were classified by Lemire-Popov-Reichstein in 2006. In 2012, the classification of stably Cayley simple groups was extended to arbitrary fields of characteristic 0 by Borovoi-Kunyavskii-Lemire-Reichstein.

On a different topic, Duncan used the classification of minimal models of rational G-surfaces to provide a classification of finite groups of essential dimension 2 over an algebraically closed field of characteristic 0. Beauville recently used Prokhorov's classification of rationally connected threefolds with an action of a simple group to classify the finite simple groups of essential dimension 3.

2.7 Motivic decompositions

Karpenko and Merkurjev gave a description of the motivic decomposition of equivariant compactifications of reduced norm varieties of central simple algebras of prime degree over a field. This result generalizes the classical motivic decompositions of norm quadrics due to Rost, as well as partial results on the case of prime degree 3 obtained by Semenov.

In a different direction, Neshitov, Petrov, Semenov and Zainoulline recently described motivic decompositions of twisted flag varieties in terms of representations of Hecke-type algebras. This creates a link between two different worlds: - the world of motives of generically split twisted flag varieties (i.e, projective homogeneous varieties under an action of a semisimple affine algebraic group such that the group splits over the function field of the variety) on one side and - the world of finitely generated projective modules over certain Hecke-type algebras on the other side.

3 Presentation Highlights and Scientific Progress Made

3.1 Alexander Vishik

In the talk of A.Vishik the idea of *anisotropy* was used to introduce the *isotropic motivic category* $\mathrm{DM}(k/k; \mathbb{F}_p)$, which is obtained from the Voevodsky category by (roughly speaking) "killing" the motives of all anisotropic (mod p) varieties over k. This "local" version of the Voevodsky category is much simpler than the "global" one and has many remarkable properties. As its global counterpart, it has the *pure part* (the analogue of the Chow motivic category) where the classical ("global") Chow groups $\mathrm{Ch} = \mathrm{CH}/p$ are substituted by their quotient $\mathrm{Ch}_{k/k}$ obtained by moding out the anisotropic classes. The notion of *flexible* fields was introduced for which the description is particularly simple, and by passing to which no information is lost. The Conjecture was formulated claiming that, over a flexible field, $\mathrm{Ch}_{k/k}$ coincides with Chow groups modulo *numerical equivalence* (mod p). And so, the "local" Chow motivic category $Chow(k/k)$ coincides with $Chow_{Num(p)}(k)$. This Conjecture was proven for varieties of dimension ≤ 5, for divisors and for cycles of dimension ≤ 2. In particularly, hypothetically (and in the above cases firmly), the "local" Chow groups are finite-dimensional, and any correspondence vanishing over algebraic closure (or even in the topological realization) is zero locally. This provides a natural environment to approach questions like *Rost Nilpotence* and the (mod p) analogues of the *Standard Conjectures*. The "local" motivic cohomology of a point (with $\mathbb{F}_2$-coefficients) over a flexible field were computed. Quite unexpectedly, these encode in themselves the Milnor's operations, which should explain, to some extent, why these operations played such a crucial role in the Voevodsky's proof of Milnor's Conjecture.

3.2 Mathieu Florence

In his talk, Mathieu Florence addressed the question of lifting vector bundles to Witt vector bundles. More precisely, let p be a prime number, and let S be a scheme of characteristic p. For any $n \geq 2$, denote by $W_n(S)$ the scheme of Witt vectors of length n, built out of S. The closed immersion $S \hookrightarrow W_n(S)$ can be thought of as a universal thickening of S, of characteristic p^n. Let V be a vector bundle over S.

Question: is V the restriction to S of a vector bundle defined over $W_n(S)$?

He gave a positive answer for line bundles: every line bundle L admits a (canonical and elementary) lift to a W_n-bundle: its n-th Witt lift. Next he considered the case of the tautological vector bundle on the base $S = \mathbb{P}_X(V)$, the projective space of a vector bundle V , defined over an affine base X. The answer is positive again. He also gave an example of a non-liftable vector bundle. Global sections of Witt lifts of line bundles can be naturally described as noteworthy algebraic objects: Pontryagin duals of divided powers of modules over Witt vectors. He ventured to explore connections between Witt vectors and divided powers. In particular, he gave a new functorial description of Witt vectors "by generators and relations", [5].

3.3 Skip Garibaldi

Let G be a simple algebraic group over an algebraically closed field k. In case $k = \mathbb{C}$, it has been known for $40+$ years which irreducible representations V of G are generically free, i.e., have the property that the stabilizer in G of a generic $v \in V$ is the trivial group scheme. Recent applications of this to the theory of essential dimension have motivated the desire to extend these results to arbitrary k. We did this in [6], [7], and [8] except for a handful of cases addressed in [9], completing the solution to the problem.

3.4 Philippe Gille

This is a report on [10]. Let K be a discretly valued henselian field with valuation ring O and residue field k. We denote by K_{nr} the maximal unramified extension of K and by K_t its maximal tamely ramified extension. If G/K is a semisimple simply connected groups, Bruhat-Tits theory is available in the sense of [13, 14] and the Galois cohomology set $H^1(K_{nr}/K, G)$ can be computed in terms of the Galois cohomology of special fibers of Bruhat-Tits group schemes. This permits to compute $H^1(K, G)$ when the residue field k is perfect. On the other hand, if k is not perfect, wild cohomology classes occur, that is $H^1(K_t, G)$ is non-trivial. Such examples appear for example in the study of bad unipotent elements of semisimple algebraic groups. Under some restrictions on G, we would like to show that $H^1(K_t/K_{nr}, G)$ vanishes (see Corollary 3.3). This is related to the following quasi-splitness result.

Theorem. Let G be a semisimple simply connected K-group which is quasi-split over K_t.

(1) If the residue field k is separably closed, then G is quasi-split.

(2) $G \times_K K_{nr}$ is quasi-split.

This theorem answers a question raised by Gopal Prasad who found another proof by reduction to the inner case of type A [14, th. 4.4]. Our first observation is that the result is quite simple to establish under the following additional hypothesis:

(*) If the variety of Borel subgroups of G carries a 0-cycle of degree one, then it has a K-rational point.

Property (*) holds away of E_8. It is an open question if (*) holds for groups of type E_8. For the E_8 case (and actually for a strongly inner K-group G) of Theorem, our proof is a Galois cohomology argument using Bruhat-Tits buildings. We can make at this stage some remarks about the statement. Since K_{nr} is a discretely valued henselian field with residue field k_s, we observe that (1) implies (2). Also a weak approximation argument reduces to the complete case. If the residue field k is separably closed of characteristic zero, we have then $cd(K) = 1$, so that the result follows from Steinbergs theorem. In other words, the main case to address is that of characteristic exponent $p > 1$.

3.5 Igor Rapinchuk

Let K be a field. The purpose of this talk was to present new results of [15] in the framework of the following general problem:

(*) (When) can one equip K with a natural set V of discrete valuations such that for a given absolutely almost simple simply connected algebraic K-group G, the set of K-isomorphism classes of (inner) K-forms of G having good reduction at all $v \in V$ (resp., at all $v \in V \setminus S$, where $S \subset V$ is an arbitrary finite subset) is finite?

While the analysis of abelian varieties defined over a global field and having good reduction at a given set of places of the field has been one of the central topics in arithmetic geometry for a long time, particularly since the work of G. Faltings [4], similar questions in various situations involving linear algebraic groups have received less attention so far.

3.6 Kirill Zainoulline

This is a report on [18]. Consider a root system Φ together with its geometric realization in $\mathbb{R}^N$, that is we look at Φ as a subset of vectors in $\mathbb{R}^N$ which is closed under reflection operators r_v for each $v \in \Phi$. A basic example of such a realization is given by the vectors

$$\Phi = \{\alpha_{ij} = e_i - e_j \in \mathbb{R}^{2n} \mid \ i \neq j,\ i,j = 1 \dots 2n,\ n \geq 1\},$$

where $\{e_1, \dots, e_{2n}\}$ is the standard basis of $\mathbb{R}^{2n}e$. This corresponds to the root system of Dynkin type A_{2n-1}. Note that $\mathbb{R}^{2n}$ admits an involutive symplectic linear operator $\tau(e_i) = -e_{2n-i+1}$ which preserves Φ. Taking averages over orbits in Φ under τ one obtains a new subset of vectors

$$\Phi_\tau = \{\frac{1}{2}(\alpha_{ij} + \tau(\alpha_{ij})) \mid \ \alpha_{ij} \in \Phi\}$$

which turns out to be a geometric realization of the root system of Dynkin type C_n. There are other similar examples of root systems (such as D_{n+1} and E_6) and involutive operators τ induced by automorphisms of the respective Dynkin diagrams. The procedure of passing from Φ to Φ_τ by taking averages via τ is called folding of a root system.

In the present paper we are weakening the assumption $\tau^2 = \mathrm{id}$ by considering an arbitrary linear automorphism $\mathcal{T}$ of $\mathbb{R}^N$ that satisfies a separable quadratic equation

$$p(\mathcal{T}) = \mathcal{T}^2 - c_1\mathcal{T} + c_2 = 0, \quad c_1, c_2 \in \mathbb{R}.$$

Hence, introducing the notion of a twisted (quadratic) folding. It would be interesting to extend our construction to affine root systems and higher degree equations.

Our motivating example is the celebrated projection of the root system of type E_8 onto the subset of icosians of the quaternion algebra which realizes a finite non-crystallographic root system of type H_4. This projection has been studied by many authors, among them by Moody-Patera [12] in the context of quasicrystals. More precisely, in this case one considers an operator $\mathcal{T}$ that satisfies the quadratic equation $x^2 - x - 1 = 0$ of the golden section and then takes the projection onto an eigenspace. We formalize such a procedure by looking at a root system $|phi$ as a subset of the Weil restriction $R_{l/k}U$ of a free module U over l where $l = k(x)/(p(x))$ is a quadratic separable algebra over an integral domain k. Then our operator $\mathcal{T}$ is the multiplication by a root τ of p that preserves the root lattice of Φ and partitions the subset of simple roots of Φ. Such data give us a folded representation of Φ. We then introduce a notion of τ-twisted folding as the projection of the root system Φ to the respective τ-eigenspace of $\mathcal{T}$. Observe that for an involution $\mathcal{T}$ this projection coincides with the usual averaging operator.

As an application we use the twisted foldings and moment graph techniques to construct maps from equivariant cohomology of flag varieties to their virtual analogue for finite Coxeter groups. Observe that this follows Sorgels philosophy to use virtual geometry to investigate combinatorics of Coxeter groups [17].

3.7 Nicole Lemire

For a central simple algebra A of degree n over a field F, the generalized Severi-Brauer variety $SB(d, A)$ is a twisted form of $Gr(d, n)$, the Grassmannian of d-dimensional planes in n-dimensional affine space. It is well-known that the variety $SB(d, A)$ has a rational point over an extension K/F if and only if $ind(A_K)|d$. We can extend this question to ask about other closed subvarieties of $SB(d, A)$. In particular, we may ask: Under what conditions does $SB(d, A)$ contain a closed subvariety which is a twisted form of a Schubert subvariety of $Gr(d, n)$? We show in [11] that this happens exactly when the index of the algebra divides a certain number arising from the combinatorics of the Schubert cell, using a variation on Fultons notion of essential set of a partition.

In the classical setting, these Schubert subvarieties are of particular interest as they form the building blocks for the Grothendieck group and Chow group of $Gr(d, n)$. The Chow groups, in the case of homogeneous varieties, and Severi-Brauer varieties in particular, have been much studied, and related to important questions about the arithmetic of central simple algebras. Although the Chow groups of dimension 0, codimension 1 and to some extent codimension 2 cycles on Severi-Brauer varieties have been amenable to study, the other groups are in general not very well understood at all. Algebraic cycles on and Chow groups of generalized Severi-Brauer varieties are even more subtle and less understood. For example, the Chow group of dimension 0 cycles on such varieties are only known in the case of reduced dimension 2 ideals in algebras of period 2. In our work we make some first steps towards developing parallel methods as currently exist for the Severi-Brauer varieties, to compute the codimension 2 Chow groups for the generalized Severi-Brauer varieties of reduced dimension 2 ideals in certain algebras of small index.

Theorem. Let $X = SB(2, A)$ with $ind(A)|12$. Then $CH^2(X)$ is torsion-free.

This computation is done first by using the explicit descriptions of Schubert classes obtained in the first half part of the paper together with other geometric constructions to show that the graded pieces of the K-groups with respect to the topological filtration are torsion free for degree 4 algebras. This quickly gives the result for codimension 2 Chow groups for such varieties. Finally, the theorem follows by an analysis of the motivic decomposition of the Chow motive of $SB(2, A)$ due to Brosnan.

3.8 Rostislav Devyatov

For details see [3].

Let G be a simple algebraic group over $\mathbb{C}$ with a simply laced Dynkin diagram. Consider the generalized flag variety G/B, where $B \subset G$ is a Borel subgroup. We are going to study the Chow ring of G/B.

The Chow ring of G/B is generated (as a $\mathbb{Z}$-algebra) by the classes of Schubert divisors in G/B (actually, to define the Schubert divisors canonically, we need to first fix a maximal torus in B, which canonically defines the root system, the Weyl group, and its action on G/B, so we assume that a maximal torus in B is fixed). Denote the classes of Schubert divisors by $D_1, \ldots, D_r$, where $r = rkG$. We will be particularly interested in monomials in classes D_i. Let us say that a monomial $D_1^{n_1} \ldots D_r^{n_r}$ is multiplicity free if there exists a Schubert class X (this is the class of a Schubert variety, not necessarily of a Schubert divisor) such that $D_1^{n_1} \ldots D_r^{n_r} X = [pt]$. Our goal is to answer the following question: What is the maximal degree (in the Chow ring) of a multiplicity-free monomial in $D_1, \dots, D_r$ (i. e., what is the maximal value of the sum $n_1 + \dots + n_r$ over all n-tuples $n_1, \dots, n_r$ of nonnegative integers such that $D_1^{n_1} \dots D_r^{n_r}$ is a multiplicity-free monomial)? This question is particularly interesting in the case when G is of type E_8, because the answer may be used to compute upper bounds on the canonical dimension of G/B.

The answer to this question for E_8 is 34. More generally, we answer this question for any simple group G with simply-laced Dynkin diagram.

3.9 Nikita Semenov

In the present talk we discuss an approach to cohomological invariants of algebraic groups over fields of characteristic zero based on the Morava K-theories, which are generalized oriented cohomology theories in the sense of Levine-Morel. We show that the second Morava K-theory detects the triviality of the Rost invariant and, more generally, relate the triviality of cohomological invariants and the splitting of Morava motives. We describe the Morava K-theory of generalized Rost motives, compute the Morava K-theory of some affine varieties, and characterize the powers of the fundamental ideal of the Witt ring with the help of the Morava K-theory. Besides, we obtain new estimates on torsion in Chow groups of codimensions up to $2n$ of quadrics from the $(n+2)$-nd power of the fundamental ideal of the Witt ring. We compute torsion in Chow groups of $K(n)$-split varieties with respect to a prime p in all codimensions up to $p^{n-1}/(p-1)$ and provide a combinatorial tool to estimate torsion up to codimension p^n. An important role in the proof is played by the gamma filtration on Morava K-theories, which gives a conceptual explanation of the nature of the torsion. Furthermore, we show that under some conditions the $K(n)$-motive of a smooth projective variety splits if and only if its $K(m)$-motive splits for all $m \leq n$.

This is a report on [16].

3.10 Maksim Zhykhovich

The classical HasseMinkowski theorem says that a non-degenerate quadratic form q over a number field F is isotropic if and only if it is isotropic over all completions of F. This assertion can be reformulated in the language of algebraic cycles and Chow motives. Namely, for the projective quadric Q given by the equation $q = 0$, the Tate motive $\mathbb{Z}(0)$ is a direct summand of the motive of Q if and only if it is a direct summand of the motive of Q_{F_v} for each completion F_v of F.

Moreover, the HasseMinkowski theorem readily implies that for every non-negative integer m, the Tate motive $\mathbb{Z}(m)$ is a direct summand of the motive of Q over F if and only if it is a direct summand of the motive of Q over all completions of F. Indeed, this is equivalent to the condition that the Witt index of q is greater than m.

We prove a generalization of the above assertion replacing the Tate motive $\mathbb{Z}(m)$ by a *binary* motive.

We say that a motive N over an arbitrary field F is a *split* motive (resp. a *binary split* motive) if it is a direct sum of a finite number of Tate motives over F (resp. if it is a direct sum of two Tate motives over F). We say that N is a binary motive over F if becomes binary split over an algebraic closure $\bar{F}$ of F.

We work in the category of Chow motives with $\mathbb{F}_2$-coefficients. We consider only non-degenerate quadratic forms. For a non-degenerate quadratic form q over a field F we denote by $M(q)$ the Chow motive of the corresponding projective quadric given by the equation $q = 0$. For a number field F and a place v of F, we denote by q_v the corresponding quadratic form over the completion F_v of F at v. Our main result is the following theorem:

Theorem, [2]. Let q be a quadratic form over a number field F. Let N be a binary split motive over F. Assume that for every place v of F there exists a direct summand ${}^vM$ of $M(q_v)$ that is isomorphic to N over an algebraic closure $\bar{F_v}$ of F_v. Then there exists a direct summand M of $M(q)$ that is isomorphic to N over F.

It follows from our proof of the above theorem that every indecomposable binary direct summand of quadric over a number field is a twist of a Rost motive. Recall that the motive of a Pfister form π over an arbitrary field is isomorphic to a direct sum of twists of one binary motive, which is called the Rost motive of π. Rost motives over an arbitrary field appear in the proof of the Milnor conjecture by Voevodsky.

We remark that over every completion of a number field, all quadratic forms are excellent, and therefore, the structure of their Chow motives is known. Moreover, every motivic decomposition of a quadric with $\mathbb{F}_2$-coefficients can be uniquely lifted to a motivic decomposition with integer coefficients. Therefore, the above theorem holds for motives with integer coefficients as well.

Finally, as an application of the Hasse principle for binary motives we obtain a complete motivic decomposition of the motive of a quadric over any number field with at most one real embedding, for example, over the field of rational numbers $\mathbb{Q}$:

Corollary. In the case when the field F has at most one real embedding, Theorem holds for every split motive N (not necessarily binary).

3.11 Nivedita Bhaskhar

Let K be a field and T, a commutative linear algebraic group defined over K. Given L/K, a finite separable field extension, one can define the *norm homomorphism* $N_{L/K} : T(L) \to T(K)$ which sends $t \rightsquigarrow \prod_\gamma \gamma(t)$ where γ runs over cosets of $\mathrm{Gal}(K^{sep}/L)$ in $\mathrm{Gal}(K^{sep}/K)$. The definition of the norm homomorphism can be extended to K-étale algebras in a similar manner. Note that if $T = \mathbb{G}_m$, then $N_{L/K} : T(L) \to T(K)$ is precisely the usual norm $N_{L/K} : L^* \to K^*$.

Now let G be a linear algebraic group defined over K and let $f : G \to T$ be an algebraic group homomorphism defined over K. We say that the *norm principle* holds for $f : G \to T$ over a finite separable field extension (or étale algebra) L/K if $N_{L/K}(\mathrm{Im} f(L)) \subseteq \mathrm{Im} f(K)$. We say that the *norm principle* holds for $f : G \to T$ if for every finite separable field extension (equivalently for every étale algebra) L/K , $N_{L/K}(\mathrm{Im} f(L)) \subseteq \mathrm{Im} f(K)$.

Suppose further that the commutator subgroup G' of G is defined over K. Then every homomorphism $f : G \to T$ factors through the natural homomorphism $\tilde{f} : G \to G/G'$ and it is an easy check that the norm principle for $\tilde{f}$ (over L/K) implies the norm principle for f (over L/K). We say that the *norm principle* holds for G (over L/K) if it holds for $\tilde{f}$ (over L/K).

Let Q be a quadratic form over K. The classical norm principle of Scharlau which asserts that norms of similarity factors of Q_L are themselves similarity factors of Q can be restated in this context to say that the norm principle holds for the multiplier map $M : \mathrm{GO}(Q) \to \mathbb{G}_m$. Similarly Knebusch's norm principle which states that norms of spinor norms of Q_L are spinor norms of Q can be reformulated as the norm principle holding for the spinor norm map $\underline{\mu} : \Gamma^+(Q) \to \mathbb{G}_m$.

Norm principles have been previously studied in especially in conjunction with the rationality or the R-triviality of the algebraic group in question. It was established that the norm principle holds in general for all reductive groups of classical type without D_n components. The D_n case was investigated later and a scalar obstruction defined up to spinor norms, whose vanishing would imply the norm principle, was given. However, the triviality of this scalar obstruction is far from clear and the question whether the norm principle holds for reductive groups with type D_n components still remains open.

If K is a number field, the norm principle was proved in full generality for all reductive groups by P. Gille , so the first widely open and very interesting case is when K is the function field $k(C)$ of a curve C defined over a number field k and the group G in question is of classical type with the semisimple part $G' = \mathrm{Spin}(Q)$. As we show, the validity of the norm principle over K is closely related to the triviality of the kernel of the natural map $H^1(K, G') \to \prod_v H^1(K_v, G')$ where v runs through a set of discrete valuations of K. Therefore the (traditional) local-global approach leads us first to look in detail over completions K_v.

With this motivation in mind, we investigate the D_n case over an arbitrary complete discretely valued field K with residue field k and $\mathrm{char}(k) \neq 2$, restricting ourselves to type D_n groups arising from quadratic forms. In the main result, we show that if the norm principle holds for such groups defined over all finite extensions of the residue field k, then it holds for such groups defined over K (see [1, Theorem 5.1]). This yields examples of complete discretely valued fields with residue fields of virtual cohomological dimension ≤ 2 over which the norm principle holds for the groups under consideration (see [1, Corollary 6.3]). As a further application, we also relate the possible failure of the norm principle to the nontriviality of certain Tate-Shaferevich sets.

4 Stefan Gille

Let R be a regular semilocal ring containing $\frac{1}{2}$ with fraction field K, and (A, τ) an R-Azumaya algebra with involution of the first or second kind. By second kind we mean that R is a quadratic Galois extension of the fix ring of the involution τ. For $\epsilon \in \{\pm 1\}$ there is a complex of ϵ-hermitian Witt groups, the so called ϵ-hermitian Gersten-Witt complex

$$0 \longrightarrow \mathrm{W}_\epsilon(A,\tau) \longrightarrow \mathrm{W}_\epsilon(A_K,\tau_K) \xrightarrow{d^{0,\epsilon}_{A,\tau}} \bigoplus_{\mathrm{ht} P=1\,,\tau(P)=P} \mathrm{W}_\epsilon(A_{k(P)},\tau_{k(P)}) \xrightarrow{d^{1,\epsilon}_{A,\tau}} \ldots\,,$$

where $k(P)$ is the residue field at the prime P of R, and where we have set $(A_{k(P)}, \tau_{k(P)}) := k(P) \otimes_R (A, \tau)$ and analogous $(A_K, \tau_K) := K \otimes_R (A, \tau)$. We consider this as a cohomological complex with the term $\mathrm{W}_\epsilon(A_K, \tau_K)$ in degree 0, and denote its cohomology groups by $\mathrm{H}^i(A, \tau, \epsilon)$.

It is known that this complex is exact if R contains a field. We extended this recently to regular semilocal rings of (very) small dimension.

Theorem A. *If R is a semilocal Dedekind domain then the ϵ-hermitian Gersten-Witt complex*

$$0 \longrightarrow \mathrm{W}_\epsilon(A,\tau) \longrightarrow \mathrm{W}_\epsilon(A_K,\tau_K) \xrightarrow{d^{0,\epsilon}_{A,\tau}} \bigoplus_{\mathrm{ht} P=1\,,\tau(P)=P} \mathrm{W}_\epsilon(A_{k(P)},\tau_{k(P)}) \longrightarrow 0$$

is exact for all $\epsilon \in \{\pm 1\}$. This sequence is split if R is a complete discrete valuation ring and τ is of the first kind.

This implies by induction that $\mathrm{H}^{\dim R}(A, \tau, \epsilon) = 0$ for all regular semilocal rings R, from which we then deduce purity for regular semilocal rings of dimension two.

Theorem B. *If the regular semilocal ring R is of dimension two then the complex*

$$0 \longrightarrow \mathrm{W}_\epsilon(A,\tau) \longrightarrow \mathrm{W}_\epsilon(A_K,\tau_K) \xrightarrow{d^{0,\epsilon}_{A,\tau}} \bigoplus_{\mathrm{ht} P=1\,,\tau(P)=P} \mathrm{W}_\epsilon(A_{k(P)},\tau_{k(P)})$$

is exact for all $\epsilon \in \{\pm 1\}$.

There are analogous exact sequences for R-orders in central simple algebras. Let B be a central simple algebra over the fraction field K of the discrete valuation ring R, and Δ a maximal R-order in B. Assume that $\frac{1}{2} \in R$, and that B has an involution of the first kind which maps Δ into itself. Then there is an exact sequence of ϵ-hermitian Witt groups ($\epsilon \in \{\pm 1\}$)

$$0 \longrightarrow \mathrm{W}_\epsilon(\Delta,\tau) \longrightarrow \mathrm{W}_\epsilon(B,\tau) \xrightarrow{\partial} \mathrm{W}_\epsilon(\Delta/\mathrm{rad}\Delta, \bar{\tau})\,,$$

where $\bar{\tau}$ is the by τ induced involution on $\Delta/\mathrm{rad}\Delta$. In case R is complete the 'residue map' ∂ is onto, and this should hold more general for arbitrary discrete valuation rings R.

5 Outcome of the Meeting

The workshop attracted 40 leading experts and young researchers from Belgium, Canada, France, South Korea, Germany, Israel, Russia, Switzerland, USA. There were 25 speakers in total: 15 talks were given by senior speakers, 9 talks by young researchers and postdocs and 2 talk by a doctoral student.

The lectures given by senior speakers provided an excellent overview on the current state of research in the theory of algebraic groups, their cohomological invariants, and motives. Young speakers were provided a unique opportunity to present their achievements. Numerous discussions between the participants after the talks have already led to several joint projects, e.g. by Auel-Suresh, Chernousov-P.Gille, Karpenko-Merkurjev.

The organizers consider the workshop to be a great success. The quantity and quality of the students, young researchers and the speakers was exceptional. The enthusiasm of the participants was evidenced by the frequent occurrence of a long line of participants waiting to ask questions to the speakers after each lecture. The organizers feel that the material these participants learned during their time in BIRS will prove to be very valuable in their research and will undoubtedly have a positive impact on the research activity in the area.

Below is a feedback of a young participant and speaker from Yale University Asher Auel.

The participants assembled was a great mix of LAG, quadratic forms, motivic, and broader arithmetic geometry people. The lectures (and choice of thematic groupings) were all great. For me, the most inspirational lectures (where I learned something new or surprising) were by Haution, Hoffmann, Bayer-Fluckiger, Garrel, Semenov, and Bhaskhar (in order of the schedule).

For me, the most important part of the workshop is the interactions.

- I had some very useful interactions with Tignol, Saltman, Duncan, Pevtsova, and Karpenko concerning the index of Weil restrictions of Severi–Brauer varieties related to a project I am working on concerning del Pezzo surfaces (one of my questions was fully resolved, and for another, Tignol pointed me to a construction in his book with Wadsworth, which helped provide an example of the phenomenon I was looking for).

- I also had some great conversations with Bayer-Fluckiger and Duncan about the fields of definitions of exceptional curves on del Pezzo surfaces, motivated by her lecture on cubic surfaces (with Duncan, we figured out the easy cases of del Pezzo surfaces of degree 5 and 6, though it's not clear if this will go anywhere until Bayer-Fluckiger and Serre make their work public).

- I had a long discussion (during the Banff airporter ride from the airport) with Lemire about her work on Schubert varieties on twisted flag varieties, where we came up with a host of interesting questions.

- After my talk, there seemed to be a flurry of attempts to prove the nontriviality of unramified cohomology in degree 3 on the product of three elliptic curves over a finite field (a major open question due to Colliot-Thelene). Both Florence and Chernousov had ideas. Actually, Florence had two ideas. His first idea seemed very promising until Parimala and Suresh found a flaw later in the day, but his second idea (using some of the theory he has developed on local systems) is still potentially viable, though I get the feeling that it will be hard to get him sufficiently interested in the questions again to flush the details out. I am sorry that I never circled back to Chernousov to inquire about his idea, so this is reminding me to write to him (which I just did). Then later in the week, Suresh and I came up with yet another strategy based on the some of the work we did together that I reported on during my talk. We are now working out the details together. If this works out, it will be great!

- However, the most sustained interaction was with Krashen. We spent the majority of Wednesday together (both of us fasting because of the Jewish holiday and therefor not hiking) working on several topics. The first was about constructing some kind of algebraic structure that explains when a central simple algebra over a degree 3 extension has trivial corestriction (analogously to an involution of the second kind over a quadratic extension). Asking Tignol, he pointed to some old work of Haile, which turned out to be relevant but not exactly what we needed. The second was about the problem of splitting Brauer classes by genus 1 curves, which I had spoke about at the last LAG meeting in Banff. I had an idea for proving several new cases, and together with Krashen, we managed to get quite a lot! So far only classes of index up to 5 are known, but we were able to prove it for all indices dividing 60. We are writing up the results now. Also, on Wednesday evening, we found ourselves discussing this question also with Hoffmann, with whom we also realized that the question of whether a quadratic form can be made hyperbolic over the function field of a genus 1 curve is also very interesting, and so likely we will write another paper with the three of us.

- I'm sure there were many other interactions of mathematical content.

- Also, after discussing with Andrei Rapinchuk, I decided to apply for a permanent position at UVA.

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