Perturbative calculations in QFT and the Laporta algorithm

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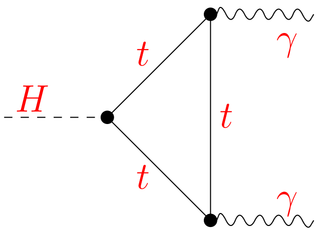
1. **Introduction: Feynman diagrams and integrals**
2. **Master Integrals (MIs) and differential equations**
3. **The Laporta algorithm and technical challenges**
4. **Solving differential equations for the MIs**
5. **Summary**

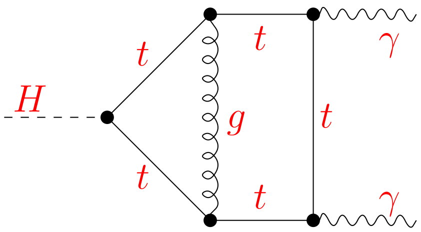
In particle physics, results of measurements are compared to theoretical predictions, most often obtained with the help of Feynman diagrams.

Examples:



Higgs boson decay to two photons

$H \to \gamma\gamma$ (t – top quark)



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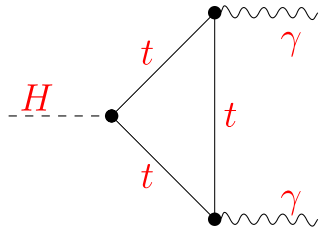

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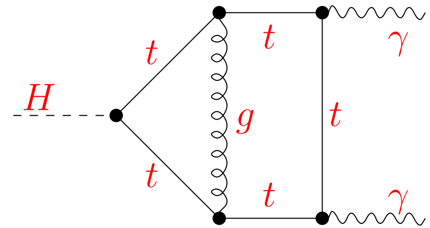

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(Sum of diagrams) $=$ (Quantum amplitude μ)

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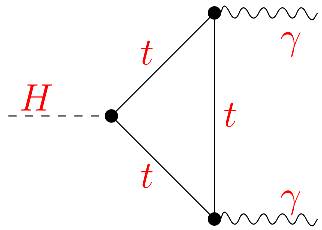

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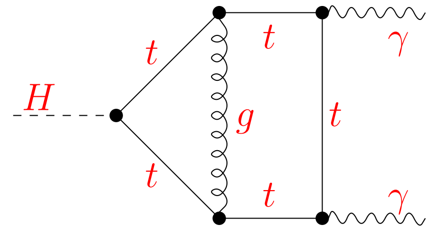

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The $\underbrace{\text{initial and final}}_{\text{external}}$ particles come with their four-momenta $p, r, \ldots$

$$p = \begin{bmatrix} E_p \\ p_x \\ p_y \\ p_z \end{bmatrix} \in \mathbb{R}^4$$

Minkowskian products of the external momenta

$$pr \equiv \mathbf{E_p E_r} - \vec{p} \cdot \vec{r} = \mathbf{E_p E_r} - (p_x r_x + p_y r_y + p_z r_z)$$

are the arguments of μ.

Each Feynman diagram with loops is specified in terms of a Feynman integrand which, after some (computer) algebra, can be written as a linear combination of expressions of the form:

$$J_{n_1 n_2 \dots n_k} = \frac{1}{A_1^{n_1} A_2^{n_2} \dots A_k^{n_k}},$$

where $n_i \in \mathbb{Z}$, and A_i are linear functions of momentum products.

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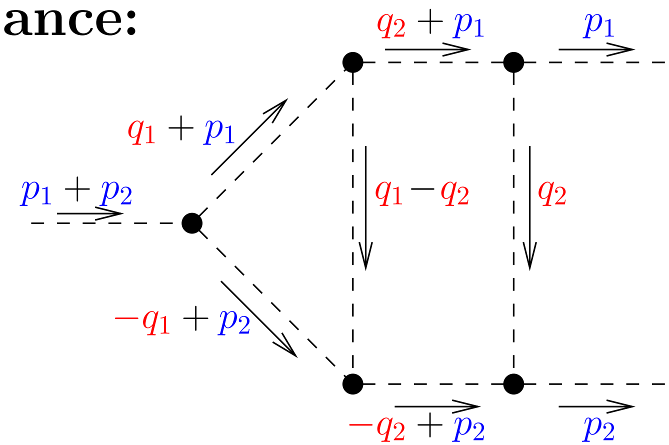

$$A_1 = M_1^2 - (q_1 + p_1)^2$$
$$A_2 = M_2^2 - (q_1 - q_2)^2$$
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$$A_6 = M_6^2 - (-q_2 + p_2)^2$$
$$A_7 = M_7^2 - q_1^2$$

$M_j^2 = m_j^2 - i\varepsilon$

$\varepsilon \in \mathbb{R}_+$, $\underbrace{m_j \in \mathbb{R}_+ \cup \{0\}}_{\text{physical masses}}$

$m_7 = 0$

Bases: $\{A_1, \ldots, A_7\} \leftrightarrow \{q_1^2, q_2^2, q_1 q_2, q_1 p_1, q_1 p_2, q_2 p_1, q_2 p_2\}$

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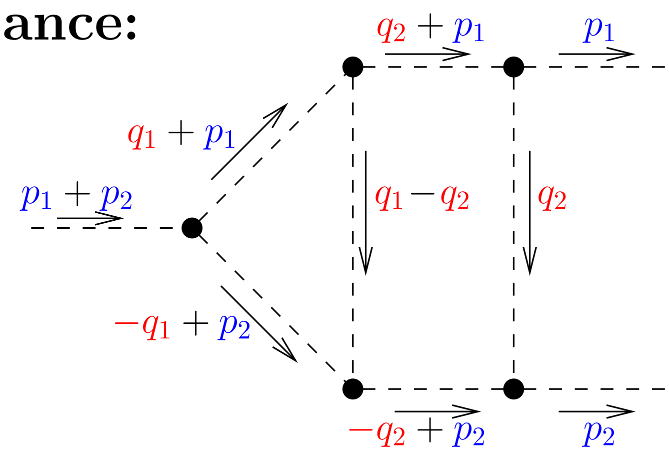

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Within the method of dimensional regularization, we find a contribution to μ by replacing

$$J_{n_1 n_2 \ldots n_k} \to I_{n_1 n_2 \ldots n_k} \equiv F[D, J_{n_1 n_2 \ldots n_k}],$$

where the r.h.s is an analytic function of $D, M_j^2 \in \mathbb{C}$ and products of external momenta.

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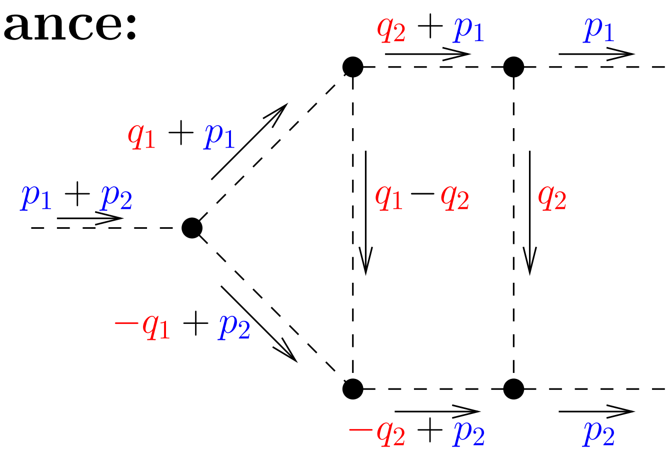

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The mapping F has the following properties:

(i) $F[D, X]$ is linear in X, while X is a rational function of momentum products and M_j^2.

(ii) $F[D, X] = 0$ when X depends neither on the external momenta nor on $m_j^2 \neq 0$.

(iii) For $D \in \mathbb{N} \setminus \{1\}$, $F[D, X] = \int (d^D q_1) \ldots (d^D q_L) X$ when the integral is finite and (ii) does not apply.

(iv) $F[D, X] = 0$ when X is a total derivative w.r.t. any of the loop momenta.

In our example $F\left[D, \frac{\partial}{\partial q_i^\alpha} (r^\alpha J)\right] = 0$, where $r \in \{q_1, q_2, p_1, p_2\}$.

Vanishing of F for total derivatives provides useful identities. Let us consider, for instance,

$$F\left[D, \frac{\partial}{\partial q_1^\alpha}\left(q_1^\alpha J_{1111110}\right)\right] = 0.$$

A straightforward calculation gives

$$\frac{\partial}{\partial q_1^\alpha}\left(q_1^\alpha J_{1111110}\right) = m^2 \left(J_{2111110} - J_{1211110} + J_{1121110}\right) +$$
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where, for simplicity, $p_1^2 = p_2^2 = m_2^2 = 0$, while all the other masses m_i have been set to m. Moreover, all the $i\varepsilon$ terms in the numerators have been tacitly set to zero.

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We can view $I_{n_1 n_2 \ldots n_k}$ as a mapping

$$I: \mathbb{Z}^k \to \mathcal{C}\left(\mathbb{C}^N\right)$$

[Complex-valued functions of $D, M_j^2 \in \mathbb{C}$ and products of external momenta (treated as complex)]

The IBP identities give us linear relations between values of I at several nearest-neighbour points. Naively, we get "more relations than integrals".

$$\boldsymbol{I} : \mathbb{Z}^{\boldsymbol{k}} \to \boldsymbol{\mathcal{C}}\left(\mathbb{C}^{N}\right)$$

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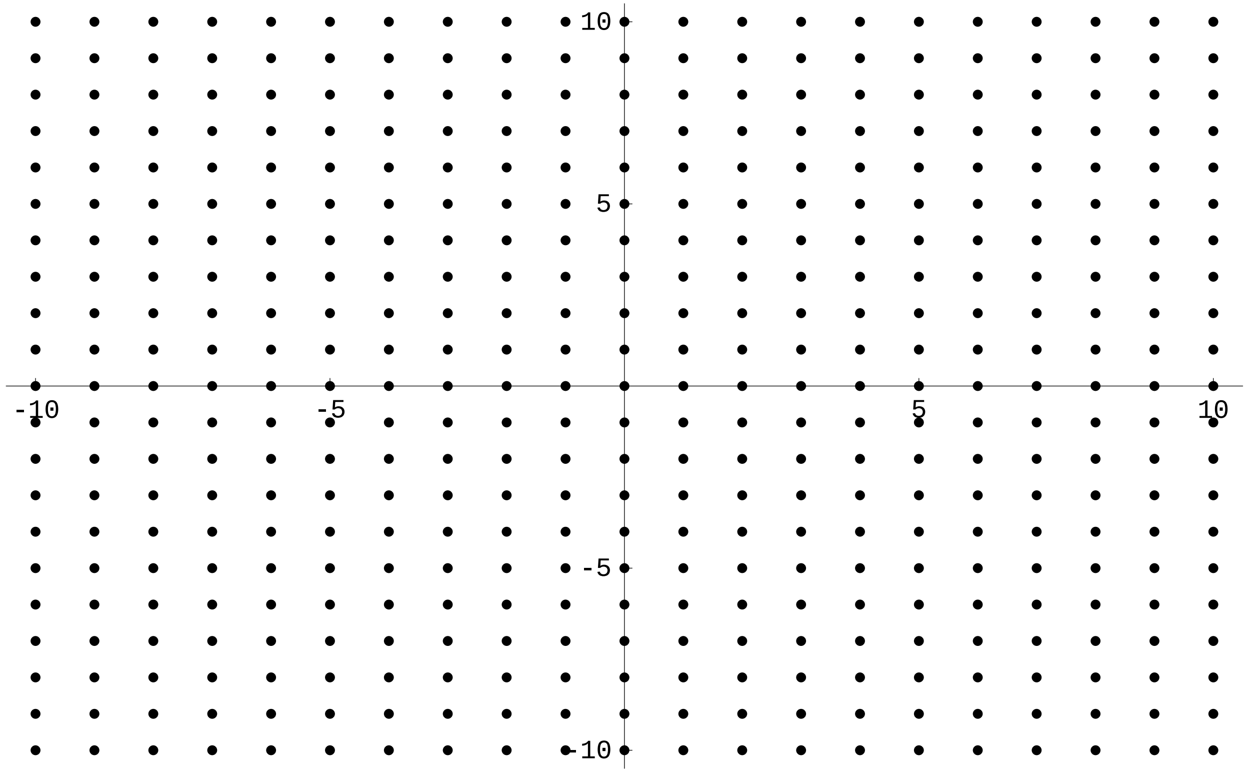

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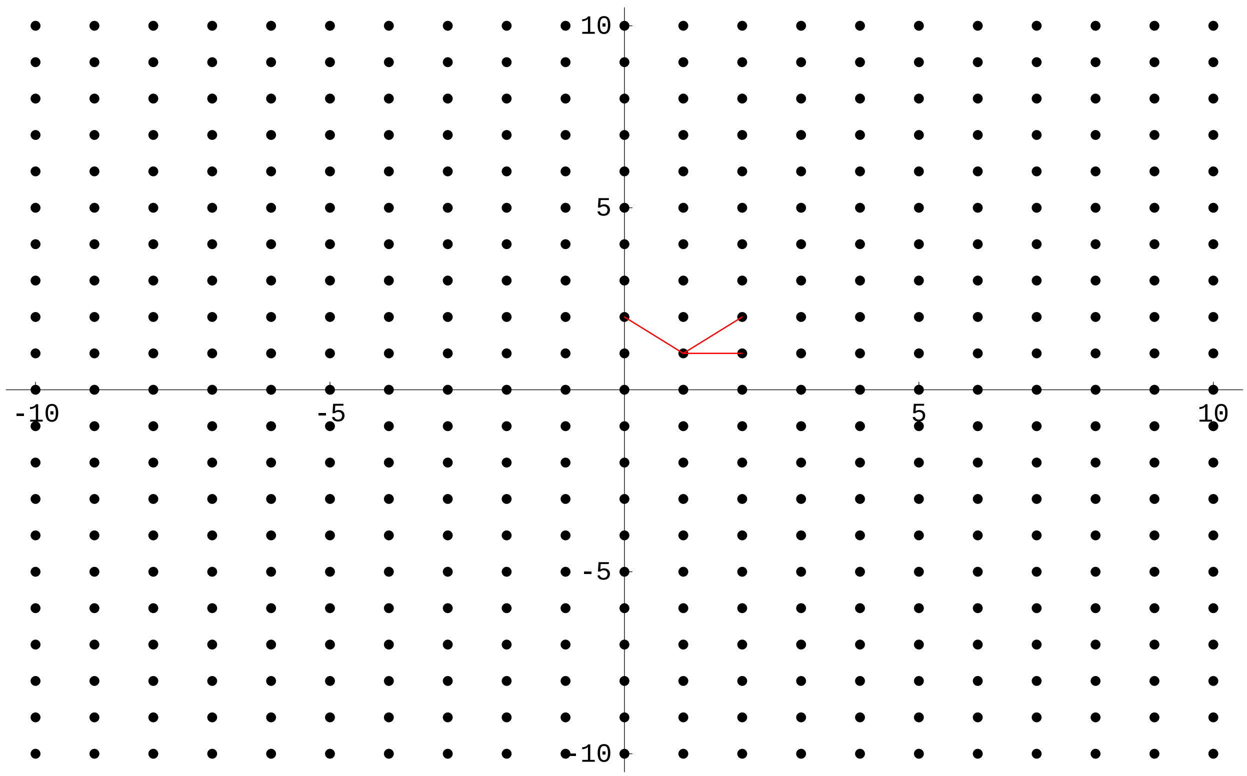

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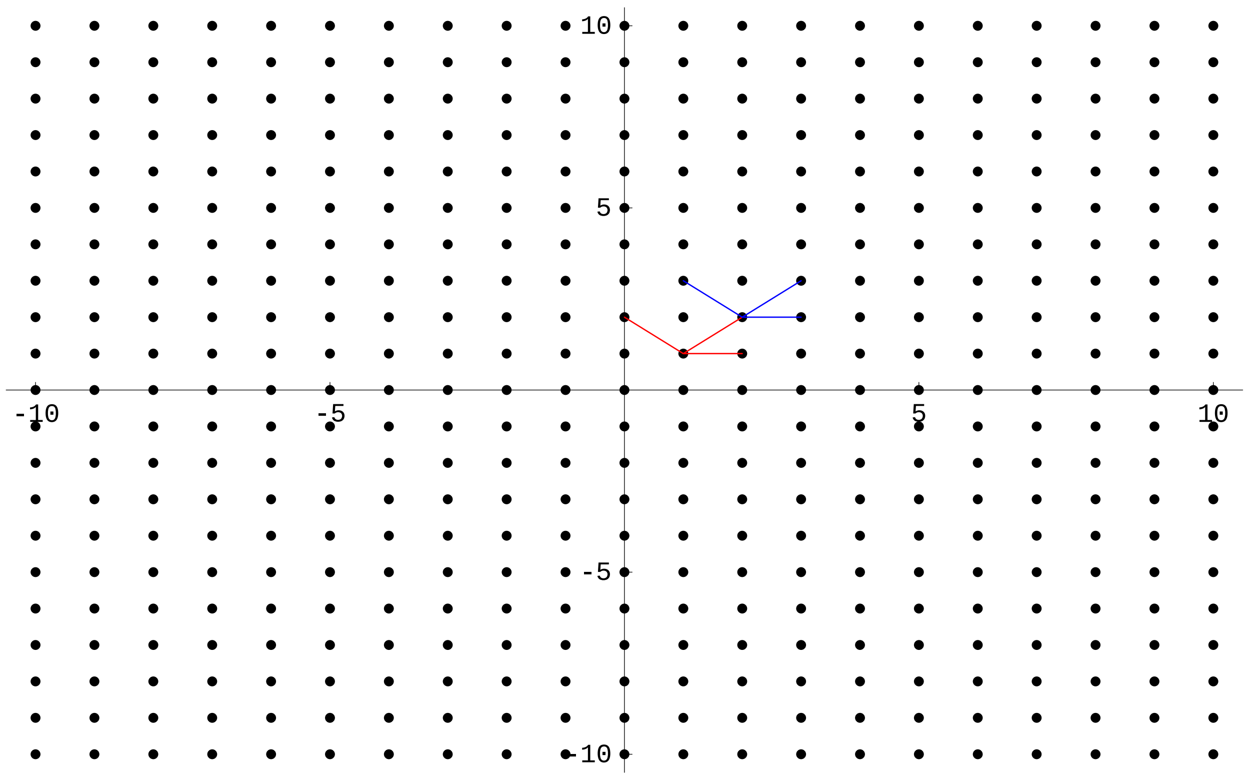

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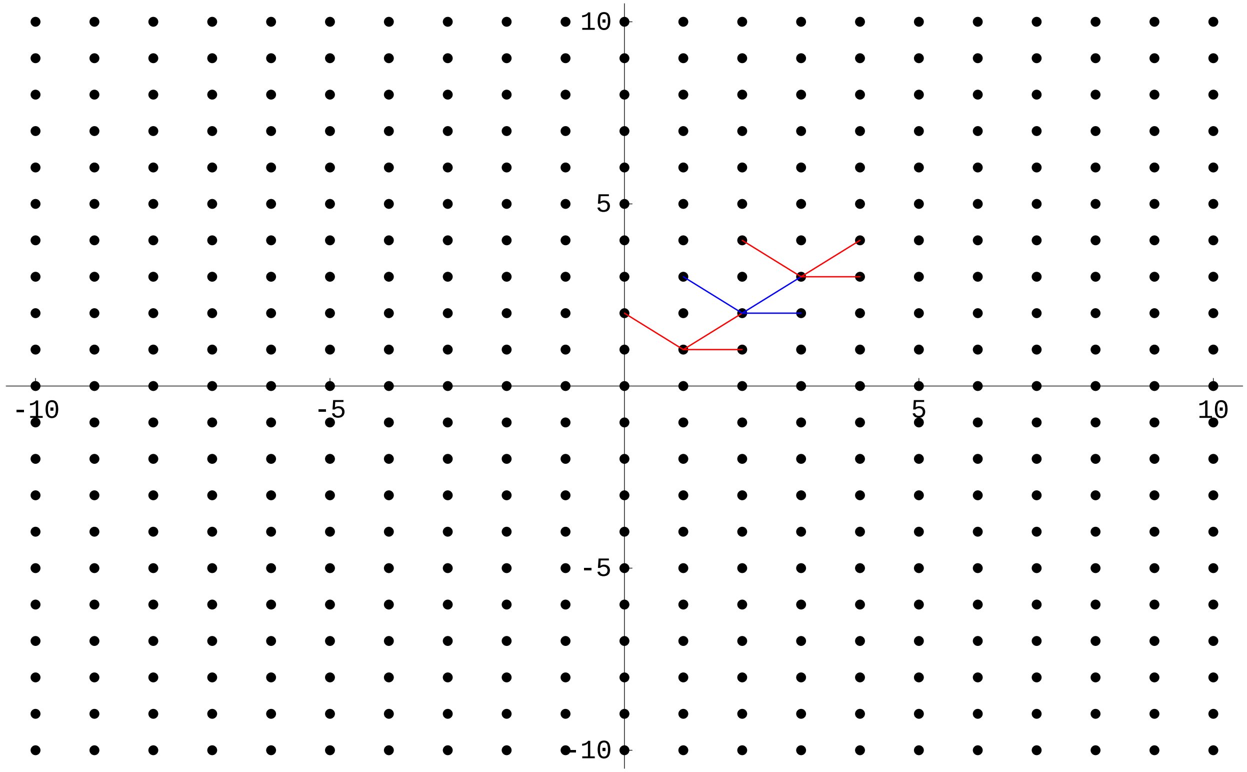

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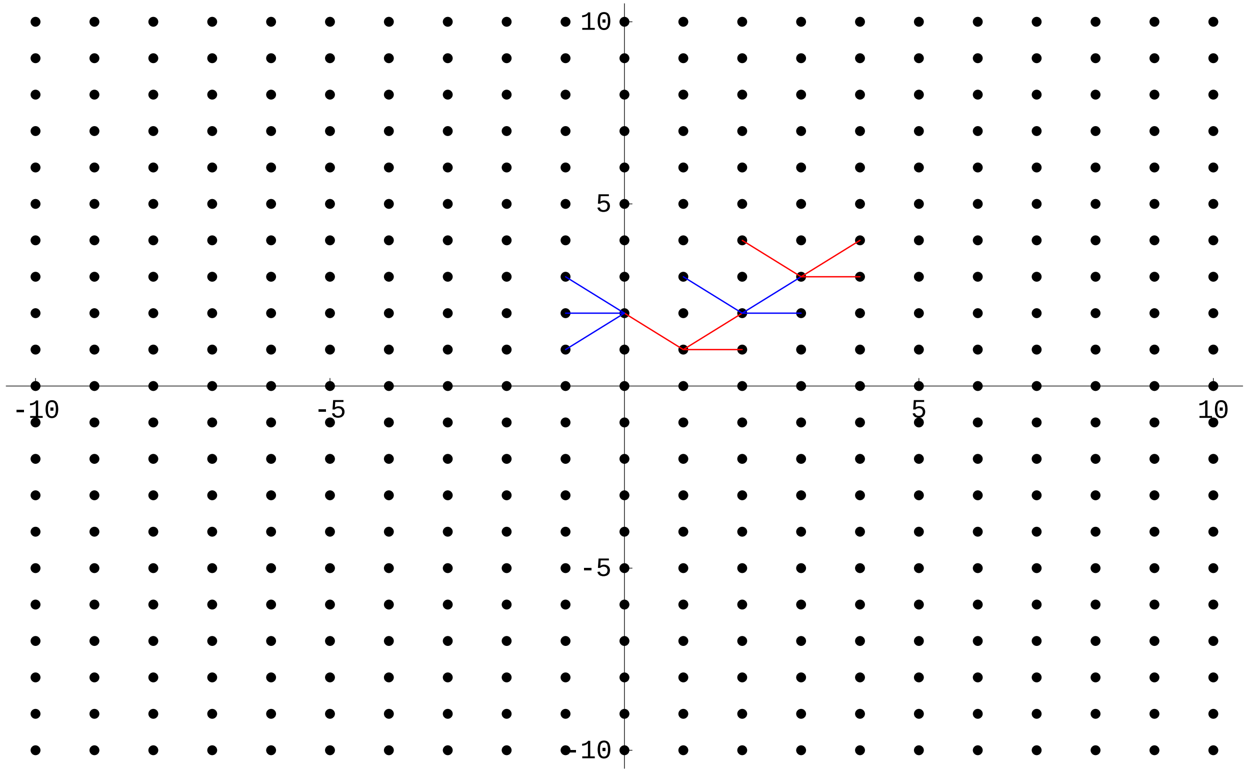

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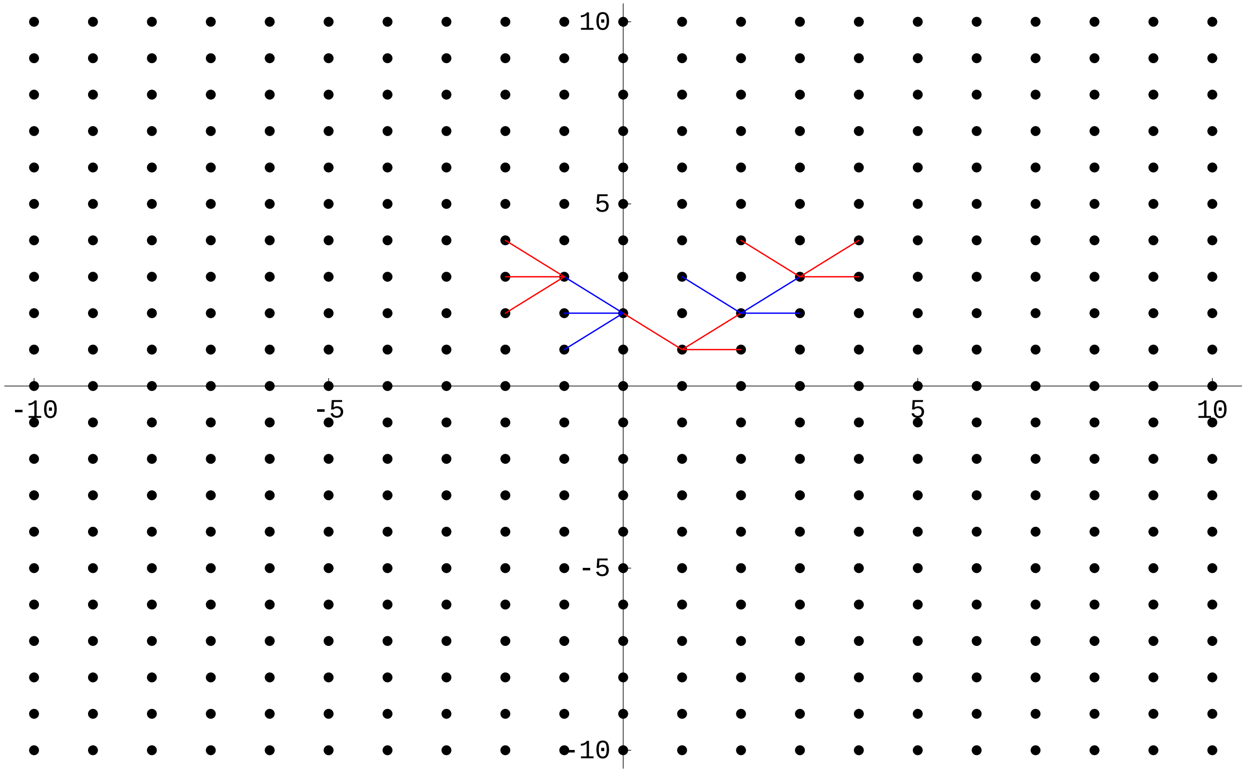

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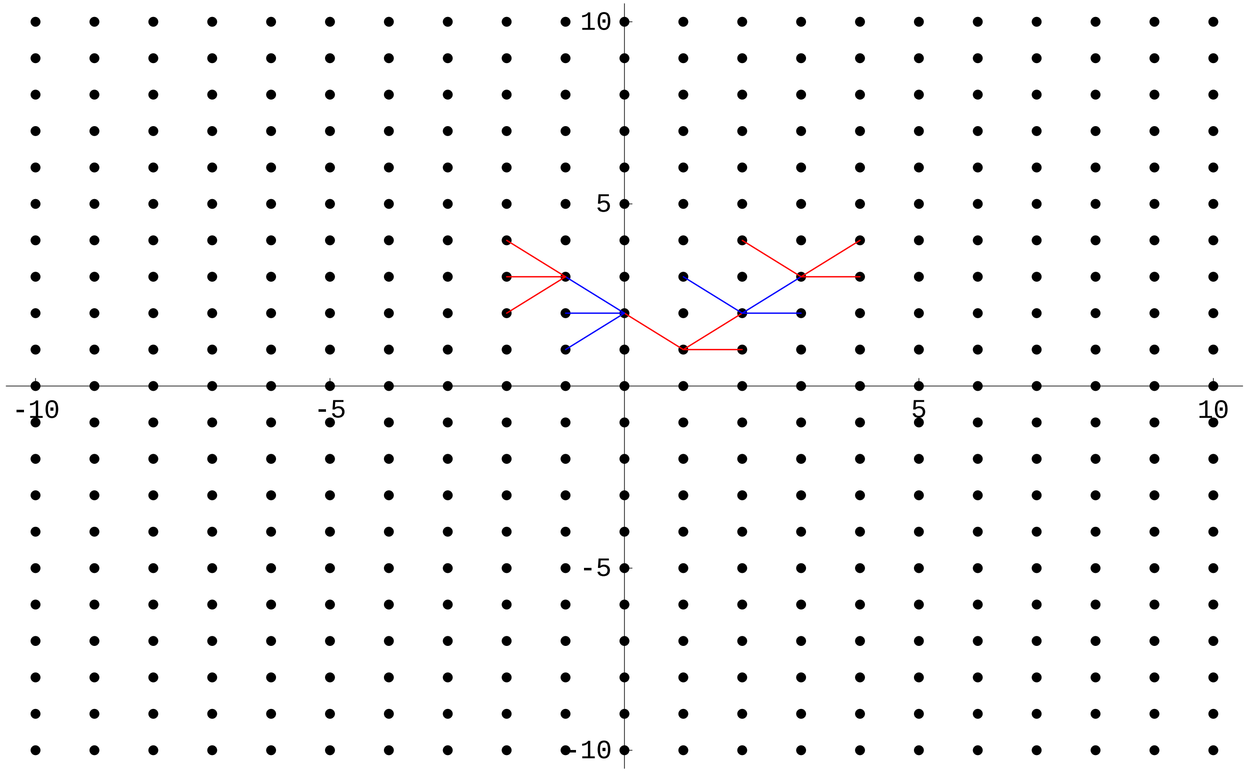

We get infinitely many linear relations among infinitely many functions.

This is similar to a linear recurrence relation, e.g., $H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z)$.

[Here we get any H_n in terms of H_0 and H_1].

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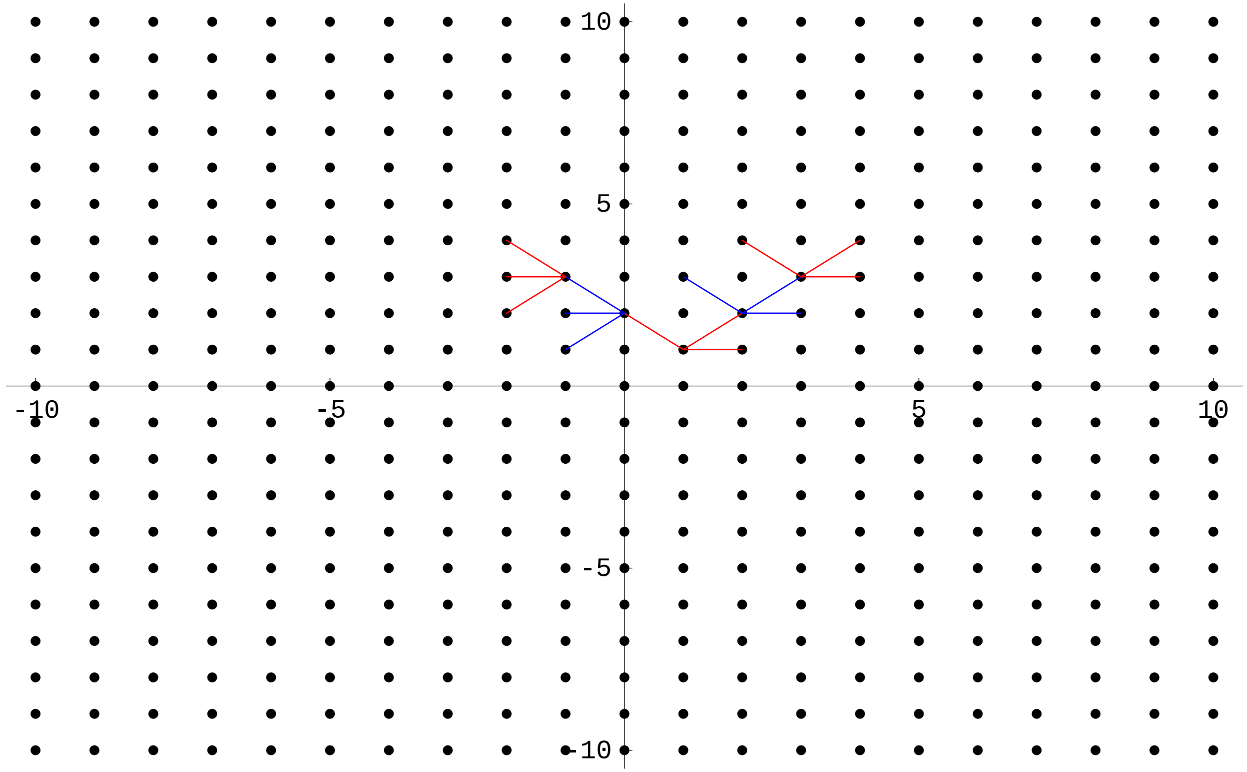

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Do the IBP give us a recurrence relation? Does a finite set of $I_{n_1 n_2 \ldots n_k}$ determine all of these functions?

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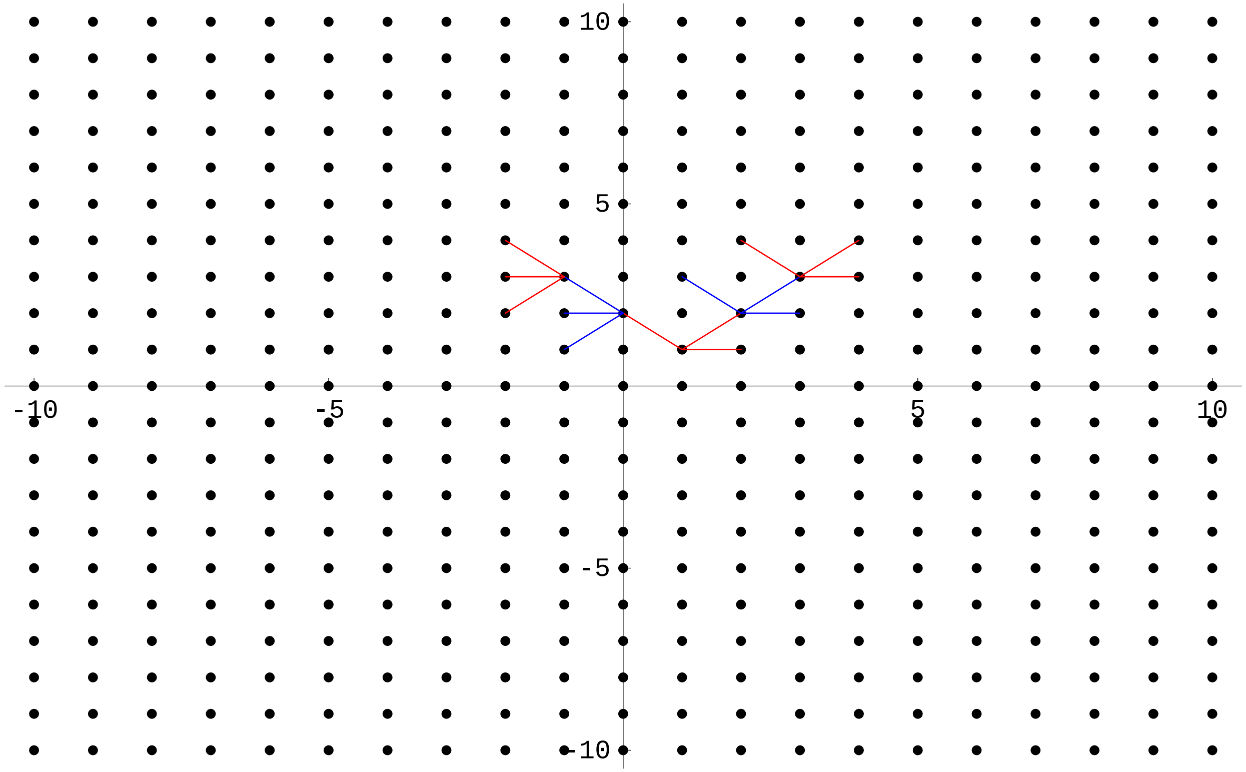

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Answer: **Yes!**

Proof: A. V. Smirnov and A. V. Petukhov, "The Number of Master Integrals is Finite," Lett. Math. Phys. 97 (2011) 37 [arXiv:1004.4199].

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In our example, when the integrand $J_{1111110}$ is differentiated w.r.t. m^2, we get:

$$\frac{\partial}{\partial(m^2)} J_{1111110} = -J_{2111110} - J_{1121110} - J_{1112110} - J_{1111210} - J_{1111120}.$$

Thus:

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Initial conditions for the DEs are set in regions where the evaluation of MIs is easier, e.g., for (masses) $\gg$ (products of external momenta).

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Consider a family of integrals $I_{n_1 \ldots n_k}$ defined by a set of denominators $\{A_1, \ldots, A_k\}$.
In any particular physical calculation only a finite set of them is relevant.

(i) We extend this set by including all the integrals with
(sum of positive indices) $\leq N_1$ and |sum of negative indices| $\leq N_2$,
with N_j fixed in a quasi-intuitive manner.

(ii) Derive all the IBP relations involving *only* the selected integrals.

(iii) Establish an absolute "simplicity" ordering in the selected set. Roughly:
First criterion: number of positive indices,
Second criterion: sum of positive indices,
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Unfortunately, this method is computationally heavy.

My current project: 451 families with $\mathcal{O}(1000)$ integrals each, depending on two variables: D and m_1/m_2.
For some families, few weeks with 1TB RAM and 2TB disk space are insufficient.

Structure of the IBP relations [see R.N. Lee, arXiv:0804.3008]

Let us consider the operators $O_{ik} = \frac{\partial}{\partial q_i} r_k$ acting on the integrands J, where $r_k \in \{q_1, \ldots, q_L, p_1, \ldots p_E\}$.
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They are related to infinitesimal redefinitions of loop momenta $q_i \to q_i' = q_i + \eta_{ik} r_k$ with constant η_{ik}, under which the integrals are invariant:

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A one-dimensional analogy:

$$f(x) \;=\; (1+\eta)\, g[(1+\eta)x + 2\eta] \quad \Rightarrow \quad \int_{-\infty}^{+\infty} f(x)dx \;=\; \int_{-\infty}^{+\infty} g(x)dx.$$

Thus, we may consider the IBP relations as following from the fact that the integrals are constant on the orbits corresponding to the generators Q_{ik}.

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This is the starting point of the proof that the number of MIs is finite.

Details in arXiv:1004.4199: “The reader familiar with the theory of holonomic D-modules should consider Theorem 2 as an exercise.”

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Algorithms and codes for solving the IBP relations based on their "group structure" have been developed. [R.N. Lee, arXiv:1212.2685, 1310.1145]. However, they are not general.

Description in terms of functions on $\mathbb{Z}^k$

[see: R.N. Lee, arXiv:0804.3008]

[more: T. Bitoun, C. Bogner, R.P. Klausen, E. Panzer, arXiv:1712:09215]

Change of notation: $\quad I_{n_1,\ldots,n_k} = f(n_1,\ldots,n_k)$

Keep the external momenta and parameters fixed, so now $\quad f : \mathbb{Z}^k \to \mathbb{C}.$

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We introduce two types of operators acting on such functions:

$$(A_\alpha f)(n_1, \ldots, n_k) = n_\alpha \, f(n_1, \ldots, n_\alpha + 1, \ldots, n_k)$$
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It is straightforward to verify that $[A_\alpha, B_\beta] = \delta_{\alpha\beta}$.

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Let $\mathcal{W}$ be the (Weyl) algebra of all possible polynomials in such operators.

Let $\mathcal{L}$ be the left ideal in $\mathcal{W}$ generated by P_{ij} : $P_{ij} f(n_1,\ldots,n_k) = F[D, O_{ij} J_{n_1,\ldots,n_k}]$.

It consists of all operators of the form $\sum_{ij} C_{ij} P_{ij}$ with $C_{ij} \in \mathcal{W}$.

All the P_{ij} have the form $a_{ij}^{\alpha\beta} A_\alpha B_\beta + b_{ij}^{\alpha} A_\alpha + c_{ij}$, with $a_{ij}^{\alpha\beta}, b_{ij}^{\alpha}, c_j \in \mathbb{C}$.

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Next, we focus on such w that $(wf)(1,\ldots,1) = f(n_1,\ldots,n_k)$ for all the indices of interest.

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To my knowledge, this problem still awaits a solution for generic $k \in \mathbb{N}$ and $a_{ij}^{\alpha\beta}, b_{ij}^\alpha, c_j \in \mathbb{C}$.

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Dedicated codes (see, e.g., arXiv:1705.06252) search (often successfully) for such matrices X that

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This is advantageous because in practice we are interested in Laurent expansions

$$\widetilde{\psi}(t,\epsilon) = \sum_{n=n_{\rm min}}^{\infty} \epsilon^n\;\widetilde{\psi}_n(t).$$

Next: Getting $\widetilde{\psi}_n(t)$ via iterative integration. Harmonic Polylogarithms (HPLs).

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- **The DEs can often be brought to a "canonical" form with the help of "gauge-like" transformations. In such cases, analytical solutions can be found via iterative integration.**