New results for the operator product expansion

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Correlation functions and the OPE

A wishlist

Correlation functions and the OPE

- We know the QFT if we know (at least) all matrix elements of all operators and their products $\langle\Psi|\hat{O}_{A_1}(x_1)\cdots\hat{O}_{A_n}(x_n)|\Psi'\rangle$

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- Various early results in perturbation theory (Zimmermann, Lowenstein, Lüscher, Mack)

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- Convergent instead of asymptotic sum

Results for scalar theories

Work of subsets of {Holland, Hollands, Kopper}

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- Factorisation: $\mathcal{C}_{A_1\cdots A_s}^{C}(x_1,\ldots,x_s) = \sum_B \mathcal{C}_{A_1\cdots A_k}^{B}(x_1,\ldots,x_k)\mathcal{C}_{BA_{k+1}\cdots A_s}^{C}(x_k,\ldots,x_s)$ holds for all $\max_{1\leq i\leq k}|x_i - x_k| < \min_{k<j\leq s}|x_i - x_k|$

Additional properties of the OPE

- Coupling constant derivative (for $g\phi^4$ interaction):
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- After suitable operator redefinitions, formulas stay valid in the massless case, but integral has IR/volume cutoff L (renormalisation scale)

Gauge theories

My work together with Holland and Hollands

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- Correlation functions are well-defined on cohomology classes: if all operators are invariant $\hat{q}\mathcal{O}_{A_k} = 0$, then for arbitrary $\mathcal{O}_B$ it follows that $\langle 0|\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_s}(x_s)|0\rangle_{\text{c,a}} = \langle 0|\mathcal{O}_{A_1}(x_1)\cdots(\mathcal{O}_{A_k} + \hat{q}\mathcal{O}_B)(x_k)\cdots\mathcal{O}_{A_s}(x_s)|0\rangle_{\text{c,a}}$

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- Take $\mathcal{O}_B = \partial_\mu \partial_\nu c$ and find all operators $\mathcal{O}_C$ on right-hand side involving $\mathcal{O}_B$ in their BRST transformation: $\mathcal{O}_{C,1} = \partial_\mu A_\nu$ and $\mathcal{O}_{C,2} = \partial_\nu A_\mu \Rightarrow \mathcal{Q}_{C,1}{}^B = \mathcal{Q}_{C,2}{}^B = 1$

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- Expansion of quantum BRST differential: $\hat{q}\mathcal{O}_A = \sum_{B:\ [\mathcal{O}_A]+1} \mathcal{Q}_A{}^B \mathcal{O}_B$
- Ward identity for OPE coefficients:
  $\sum_{k=1}^{s} \sum_{C:\ [\mathcal{O}_C] \leq [\mathcal{O}_{A_k}]+1} \mathcal{Q}_{A_k}{}^C \mathcal{C}^B_{A_1 \cdots A_{k-1} C A_{k+1} \cdots A_s}(\mathbf{x}) = \sum_C \mathcal{Q}_C{}^B \mathcal{C}^C_{A_1 \cdots A_s}(\mathbf{x})$
- Example: free electromagnetism $\hat{s}_0 A_\mu = \partial_\mu c$, $\hat{s}_0 c = 0$ and all operators invariant $\hat{s}_0 \mathcal{O}_{A_k} = 0$ such that $\mathcal{Q}_{A_k}{}^C = 0$ (e.g. $\mathcal{O} = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mathcal{O} = F_{\mu\nu} F^{\nu\rho}, \ldots$)
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- Ward identity $\mathcal{C}^{\partial_\mu A_\nu}_{A_1 \cdots A_s}(x_1, \ldots, x_s) + \mathcal{C}^{\partial_\nu A_\mu}_{A_1 \cdots A_s}(x_1, \ldots, x_s) = 0$ and OPE
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- In the OPE of $\hat{q}$-invariant operators only $\hat{q}$-invariant operators appear on the right-hand side (would follow automatically for convergent expansion)

OPE properties in massless theories

- OPE is (at least) asymptotic: $\lim_{\tau\to 0} \tau^{[\mathcal{O}_{\mathbf{A}}]-D+\delta} \Big[\langle\Psi|\mathcal{O}_{A_1}(\tau x_1)\cdots\mathcal{O}_{A_s}(\tau x_s)|\Psi\rangle - \sum_{B:\,[\mathcal{O}_B]<D} \mathcal{C}^{B}_{A_1\cdots A_s}(\tau\mathbf{x})\langle\Psi|\mathcal{O}_B(\tau x_s)|\Psi\rangle \Big] = 0$ for all $\delta > 0$

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- Factorisation: $\mathcal{C}^C_{A_1\cdots A_n}(x_1,\ldots,x_n) = \sum_B \mathcal{C}^B_{A_1\cdots A_k}(x_1,\ldots,x_k)\mathcal{C}^C_{BA_{k+1}\cdots A_n}(x_k,\ldots,x_n)$ holds for all $\max_{1\le i\le k}|x_i - x_k| < \min_{k<j\le n}|x_i - x_k|$

Recursive constructions

- Coupling constant derivative: $\partial_g \mathcal{C}^B_{A_1 \cdots A_s}(\mathbf{x}) =$
$\int \sum_{E:\, 1 \le [\mathcal{O}_E] \le 4} \mathcal{I}^E \Big[-\mathcal{C}^B_{E A_1 \cdots A_s}(y, \mathbf{x}) + \sum_{C:\, [\mathcal{O}_C] < [\mathcal{O}_B]} \mathcal{C}^C_{A_1 \cdots A_s}(\mathbf{x}) \mathcal{C}^B_{EC}(y, x_s) +$
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$\sum_{C:\ [\mathcal{O}_B]\le[\mathcal{O}_C]\le[\mathcal{O}_A]+1} \mathcal{Q}_A{}^C \mathcal{C}^{B}_{EC}(y,0)\Big]\, \mathrm{d}^4 y$

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$\sum_{C:\ [\mathcal{O}_B] \leq [\mathcal{O}_C] \leq [\mathcal{O}_A]+1} \mathcal{Q}_A{}^C \mathcal{C}^B_{EC}(y, 0) \Big] \, \mathrm{d}^4 y$
- Interaction operator $\mathcal{O}_{\mathcal{I}} = \sum_{E:\ 1 \leq [\mathcal{O}_E] \leq 4} \mathcal{I}^E \mathcal{O}_E$ with $\mathcal{O}_{\mathcal{I}} = \partial_g L|_{g=0} + \mathcal{O}(g) + \mathcal{O}(\hbar)$ and $\hat{q} \mathcal{O}_{\mathcal{I}} = \mathrm{d}\mathcal{O}'$ for some $\mathcal{O}'$

Thank you for your attention

Questions?

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