The resolvent algebra of non-relativistic Bose fields: a C*-dynamical approach to interacting many body systems

Detlev Buchholz

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Motivation

Many body theory for Bosons: Bose fields

- annihilation and creation operators, **complex** $f, g \in \mathcal{D}(\mathbb{R}^s)$,

$$[a(f), a^*(g)] = \langle f, g \rangle \, 1 \,, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0 \,.$$

- fields (**real** linear)

$$\phi(f) \doteq a^*(f) + a(f) \,.$$

- Fock space $\mathcal{F}$: generated by fields from vacuum Ω

- dynamics: (pair potentials $V \in C_0(\mathbb{R}^s)$)

$$\boldsymbol{H} = \int d\boldsymbol{x} \, \partial a^*(\boldsymbol{x}) \partial a(\boldsymbol{x}) + \int d\boldsymbol{x} \int d\boldsymbol{y} \; a^*(\boldsymbol{x}) a^*(\boldsymbol{y}) V(\boldsymbol{x} - \boldsymbol{y}) a(\boldsymbol{x}) a(\boldsymbol{y})$$

Standard framework (“bookkeeping”). Other (*e.g.* thermal) states require different representations and modified Hamiltonians . . .

Longstanding question:

Does there exist a "kinematical" C*-algebra $\mathfrak{A}$, encoding the CCRs, such that the solutions of the Heisenberg equation lie in $\mathfrak{A}$, *i.e.*

$$\partial_t A(t) = i\,[\boldsymbol{H}, A(t)]\,,\ A(0) \in \mathfrak{A}_{\text{o}} \ \text{ implies } \ A(t) \in \mathfrak{A}_{\text{o}},\ t \in \mathbb{R}\,?$$

Consequence: $A(t) = \alpha(t)(A)$, where $\alpha(t)$ are automorphisms of $\mathfrak{A}$, $t \in \mathbb{R}$.

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Sceptical views: [Bratteli, Robinson]

> First, no global statement of the time development as a group of *-automorphisms of an appropriate C^*-algebra has been obtained and, second, there are plausible physical reasons for believing that such a development, if it existed, would be discontinuous in the norm topology.

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Sceptical views: [Narnhofer, Thirring]

> The real trouble maker is $\rho(x)$ which in some representation like the Fock representation is well defined but in others, like the one based on the tracial state, is truly infinite. Though there is no doubt that for a smooth v H from (1.2) determines a time evolution in the Fock representation, an automorphism of the algebra of observables, which would be valid in any state, cannot be expected in general.

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Conclusion: "large field problems" obstruct C*-algebraic approach ...

Outline

- Resolvent algebra
- Gauge transformations
- Structure of observables
- Dynamics of observables and fields
- Applications
- Summary

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Resolvent algebra

CCRs can be encoded in relations between resolvents of field,

$$R(\lambda, f) = (i\lambda + \phi(f))^{-1}, \quad \lambda \in \mathbb{R}\backslash\{0\}, \ f \in \mathcal{D}(\mathbb{R}^s)$$

1. $R(\lambda, 0) = (i\lambda)^{-1} 1$
2. $R(\lambda, f)^* = R(-\lambda, f)$
3. $R(\lambda, f) - R(\mu, f) = i(\mu - \lambda)R(\lambda, f)R(\mu, f)$
4. $[R(\lambda, f), R(\mu, g)] = i\sigma(f, g) R(\lambda, f) R(\mu, g)^2 R(\lambda, f)$
5. $\nu R(\nu\lambda, \nu f) = R(\lambda, f)$
6. $R(\lambda, f)R(\mu, g)$
   $= R(\lambda + \mu, f + g)[R(\lambda, f) + R(\mu, g) + i\sigma(f, g)R(\lambda, f)^2 R(\mu, g)]$

Resolvent algebra: abstract C*-algebra $\mathfrak{R}$ generated by all sums and products of these symbols. Faithfully represented on $\mathcal{F}$.

Gauge transformations

On $\mathfrak{R}$ acts global gauge group $\Gamma \simeq U(1)$ given by

$$\gamma(u)(R(\lambda, f)) \doteq R(\lambda, e^{iu} f) \stackrel{\mathcal{F}}{=} e^{iuN} R(\lambda, f) e^{-iuN}, \quad u \in [0, 2\pi],$$

N particle number operator.

Action of gauge transformations discontinuous in C*-sense. Nevertheless

Lemma

Let $R \in \mathfrak{R}$. Its Fourier components are elements of $\mathfrak{R}$, i.e.

$$R_m \stackrel{\mathcal{F}}{=} (1/2\pi)^{-1} \int_0^{2\pi} du\, e^{imu}\, \gamma(u)(R) \in \mathfrak{R}, \quad m \in \mathbb{Z}.$$

Note: integral is defined only in the strong operator topology on $\mathcal{F}$.

Observable algebra: $\mathfrak{A} = \mathfrak{R}^\Gamma \subset \mathfrak{R}$ (gauge invariant elements).

Outline of proof:

Let $f \in \mathcal{D}(\mathbb{R}^s)$ and put (i) $L \doteq \mathbb{C}\, f$, (ii) $\mathcal{F}(L) \subset \mathcal{F}$ Fock space over L, (iii) $\mathfrak{R}(L) \subset \mathfrak{R}$. Consider

$$u, v \mapsto e^{im(u-v)} \gamma(v)(R(\lambda, f))^* \, \gamma(u)(R(\lambda, f)) = e^{im(u-v)} R(-\lambda, e^{iv} f) \, R(\lambda, e^{iu} f) \in \mathfrak{R}(L) \, .$$

Underlying field operators satisfy

$$[\phi(e^{iu} f), \phi(e^{iv} f)] = (e^{i(v-u)} - e^{-i(v-u)}) \, \langle f, f \rangle \neq 0 \quad \text{if} \quad (u - v) \neq \pi \mathbb{Z} \, .$$

Operator function has values in ideal $\mathfrak{C}(L) \subset \mathfrak{R}(L)$ of compact operators on $\mathcal{F}(L)$ for almost all $(u, v) \in \mathbb{R}^2$ and it is bounded. Hence (s.o. topology)

$$\int_0^{2\pi} dv \int_0^{2\pi} du \, e^{im(u-v)} \gamma(v)(R(\lambda, f))^* \, \gamma(u)(R(\lambda, f)) = \Big| \int_0^{2\pi} du \, e^{imu} \gamma(u)(R(\lambda, f)) \Big|^2 \in \mathfrak{C}(L) \, .$$

Polar decomposition: $\int_0^{2\pi} du \, e^{imu} \gamma(u)(R(\lambda, f)) \in \mathfrak{C}(L) \subset \mathfrak{R}$.

Structure of observables

Detailed analysis necessary. Basic facts:

- $\mathfrak{A}$ acts faithfully on Fock space $\mathcal{F} = \oplus_n \mathcal{F}_n$ (since $\mathfrak{R}$ does)
- $\rho_n(\mathfrak{A}) \doteq \mathfrak{A} \upharpoonright \mathcal{F}_n$ disjoint, non-faithful representations of $\mathfrak{A}$, $n \in \mathbb{N}_0$

Strategy of analysis:

- clarify structure of each $\rho_n(\mathfrak{A})$
- understand relation between different algebras $\rho_n(\mathfrak{A})$, $n \in \mathbb{N}_0$

Definition: $\mathfrak{C}_m$ compact operators on $\mathcal{F}_m$. Natural embedding into $\mathcal{F}_n$

$$\mathfrak{C}_m \mapsto \mathfrak{C}_{m\,n} \doteq \mathfrak{C}_m \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-m}, \quad 0 \leq m \leq n$$

$\mathfrak{K}_n \doteq$ linear span of $\mathfrak{C}_{m\,n}$, $0 \leq m \leq n$ (AF algebra).

Proposition

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Let $n \in \mathbb{N}_0$, then $\rho_n(\mathfrak{A}) = \mathfrak{K}_n$.

Definition: $\{\mathfrak{K}_n, \epsilon_n\}_{n\in\mathbb{N}_0}$ where $\epsilon_n(\mathfrak{K}_n) \doteq \mathfrak{K}_n \otimes_s 1 \subset \mathfrak{K}_{n+1}$ (directed system)

Relation between $\mathfrak{K}_n = \rho_n(\mathfrak{A})$, $n \in \mathbb{N}_0$? Use clustering properties!

$$\boldsymbol{\Phi}^n(\boldsymbol{x}) \doteq \underbrace{\Phi_1 \otimes_s \cdots \otimes_s \Phi_{n-1}}_{\boldsymbol{\Phi}^{n-1}} \otimes_s \Phi_n(\boldsymbol{x}) \in \mathcal{F}_n\,, \quad \Phi_1, \dots, \Phi_n \in \mathcal{F}_1\,,\ \boldsymbol{x} \in \mathbb{R}^s\,.$$

Proposition

Let $n \in \mathbb{N}$, $A \in \mathfrak{A}$.

(i) $\lim_{\boldsymbol{x}\to\infty} \langle \boldsymbol{\Psi}^n(\boldsymbol{x}), \rho_n(A)\, \boldsymbol{\Phi}^n(\boldsymbol{x})\rangle = n^{-1} \langle \boldsymbol{\Psi}^{n-1}, \rho_{n-1}(A)\, \boldsymbol{\Phi}^{n-1}\rangle \langle \Psi_n, \Phi_n\rangle$

(ii) $\rho_n(A) = \sum_{m=0}^n C_{m\,n}$ *implies* $\rho_{n-1}(A) = \sum_{m=0}^n \frac{n-m}{n}\, C_{m\,n-1}$

Recall notation: $C_{k\,l} = C_k \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{l-k} \in \mathfrak{C}_{k\,l} \subset \mathfrak{K}_l$.

Definition: $\{\mathfrak{K}_n, \kappa_n\}_{n\in\mathbb{N}_0}$ where $\kappa_n : \mathfrak{K}_n \to \mathfrak{K}_{n-1}$ homomorphism given by $\kappa_n(\sum_{m=0}^n C_{m\,n}) = \sum_{m=0}^n \frac{n-m}{n}\, C_{m\,n-1}$ (inverse system)

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Remark: C*-algebra, algebraic operations component-wise defined. Proposition implies $\mathfrak{A} \subset \mathfrak{K}$. Extend $\mathfrak{A}$ in order to obtain equality!

Definition: $\overline{\mathfrak{A}}$ is defined as the C*-algebra of all bounded operators A on $\mathcal{F}$ satisfying $A \upharpoonright \oplus_{m=0}^{n} \mathcal{F}_m \in \mathfrak{A} \upharpoonright \oplus_{m=0}^{n} \mathcal{F}_m$, $n \in \mathbb{N}_0$.

Remark: $\mathfrak{A}$ dense in $\overline{\mathfrak{A}}$ with regard to the locally convex topology induced by seminorms $\| \cdot \|_n$, $n \in \mathbb{N}_0$. Differences between algebras only visible in states containing an infinity of particles.

Theorem

Map $A \mapsto \{A \upharpoonright \mathcal{F}_n \in \mathfrak{K}_n\}_{n \in \mathbb{N}_0}$ defines isomorphism between $\overline{\mathfrak{A}}$ and $\mathfrak{K}$.

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Dynamics of observables and fields

Strategy:

- establish stability of $\mathfrak{K}_n$ under action of dynamics, $n \in \mathbb{N}_0$
- check consistency of dynamics with coherence condition

Analysis:

- Consider restrictions $\boldsymbol{H} \upharpoonright \mathcal{F}_n = H_n,\ n \in \mathbb{N}_0,$ where

$$H_n = \sum_i \boldsymbol{P}_i^2 + \sum_{j \neq k} V(\boldsymbol{Q}_j - \boldsymbol{Q}_k)\,, \quad i,j,k \in \{1,\dots,n\}.$$

  Define automorphic action of dynamics on $\mathcal{B}(\mathcal{F}_n)$

$$\alpha_n(t) \doteq \mathrm{Ad}\, e^{itH_n}\,, \quad t \in \mathbb{R}\,.$$

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Note: $\mathfrak{K}_n$ has ideals; result not true for simple subalgebras of $\mathcal{B}(\mathcal{F}_n)$.

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$\kappa_n \circ \alpha_n(t) = \alpha_{n-1}(t) \circ \kappa_n$ on $\mathfrak{K}_n$, $n \in \mathbb{N}_0$.

Consequence: $\{K_n\}_{n \in \mathbb{N}_0} \in \mathfrak{K}$ implies $\{\alpha_n(t)(K_n)\}_{n \in \mathbb{N}_0} \in \mathfrak{K}$, $t \in \mathbb{R}$.

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Let $\alpha(t)$, $t \in \mathbb{R}$, be the group of automorphisms of $\mathcal{B}(\mathcal{F})$ fixed by a Hamiltonian $\boldsymbol{H}$ with pair potential $V \in C_0(\mathbb{R}^s)$.

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(ii) There is a dense (in l.c.t.) subalgebra $\overline{\mathfrak{A}}_\alpha \subset \overline{\mathfrak{A}}$ on which action is pointwise norm continuous, i.e. $(\overline{\mathfrak{A}}_\alpha, \alpha)$ is a C^*-dynamical system.

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Action of dynamics on non-gauge invariant operators (fields)

Definition: $V_f \doteq a^*(f)(1 + a^*(f)a(f))^{-1/2}$, $f \in \mathcal{D}(\mathbb{R}^s)$ normalized

Facts:

- $V_f^* V_f = 1$, $V_f V_f^* = E_f$ (projection onto $\ker a(f)^\perp$)
- $\sigma_f(\,\cdot\,) \doteq V_f \cdot V_f^*$ defines morphism of $\overline{\mathfrak{A}}$ (non-unital)
- $V_g V_f^*$, $V_f V_g^* \in \overline{\mathfrak{A}}$ (transportability of morphisms, $\rho_f \mapsto \rho_g$).

Action of dynamics (defined on $\mathcal{F}$):

- $\alpha(t)(V_f) = \big(\alpha(t)(V_f)\,V_f^*\big)\,V_f, \quad \alpha(t)(V_f^*) = V_f^*\big(V_f\,\alpha(t)(V_f^*)\big),$

Proposition

Let $\alpha(t)$, $t \in \mathbb{R}$, be defined as above and pick normalized $f \in \mathcal{D}(\mathbb{R}^s)$.

(i) $\alpha(t)(V_f)V_f^*$, $V_f\,\alpha(t)(V_f^*) \in \overline{\mathfrak{A}}$, $t \in \mathbb{R}$.

(ii) *C^*-algebra $\overline{\mathfrak{R}}$ generated by $\overline{\mathfrak{A}}$ and V_f, V_f^* is stable under the automorphic action of $\alpha(t)$, $t \in \mathbb{R}$.*

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Applications

1. Quasi local structure of observables
2. (Approximate) ground states and condensates
3. Particle properties and collision theory
4. Equilibrium states

(1) Quasi local structure of observables

Consider spatial translations, fixed by $\alpha_{\boldsymbol{x}}(R(\lambda, f)) = R(\lambda, f_{\boldsymbol{x}})$, and put $\boldsymbol{\alpha}(t, \boldsymbol{x}) \doteq \boldsymbol{\alpha}(t) \circ \boldsymbol{\alpha}(\boldsymbol{x}) = \boldsymbol{\alpha}(\boldsymbol{x}) \circ \boldsymbol{\alpha}(t)$ for $(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^s$

Corollary

Let $A, B \in \overline{\mathfrak{A}}$ *and* $t \in \mathbb{R}$. *Then*

$$\lim_{\boldsymbol{x} \to \infty} \| [\alpha(t, \boldsymbol{x})(A), B] \|_n = 0 \, , \quad n \in \mathbb{N}_0 \, .$$

Question: Do there hold more specific bounds for given potential V?

(2) (Approximate) ground states and condensates

(a) Ground state: Ω for (renormalized) Hamiltonian $\boldsymbol{H}_r = \boldsymbol{H} + E(N)$.

(b) Approximate (non-Fock) ground states and condensates:

$$\boldsymbol{\Psi}_{L,n} = (n!)^{-1/2}\, \Phi_L \otimes_s \cdots \otimes_s \Phi_L \in \mathcal{F}_n\,, \quad n \in \mathbb{N}$$

where $\boldsymbol{x} \mapsto \Phi_L(\boldsymbol{x}) \doteq L^{-s/2}\, \Phi(\boldsymbol{x}/L) \in \mathcal{F}_1$ is normalized.

Let $V \geq 0$, $\widehat{\Omega}_n$ outgoing Møller operator, $\widehat{\boldsymbol{\Psi}}_{L,n} \doteq \widehat{\Omega}_n \boldsymbol{\Psi}_{L,n}$, then

$$0 \leq \langle \widehat{\boldsymbol{\Psi}}_{L,n}, H_n \widehat{\boldsymbol{\Psi}}_{L,n} \rangle = \langle \boldsymbol{\Psi}_{L,n}, H_{0,n} \boldsymbol{\Psi}_{L,n} \rangle = nL^{-2} \int d\boldsymbol{x}\, |\partial \Phi(\boldsymbol{x})|^2$$

Consider states $\omega_{n,L}(\,\cdot\,) \doteq \langle \widehat{\boldsymbol{\Psi}}_{L,n}, \,\cdot\, \widehat{\boldsymbol{\Psi}}_{L,n} \rangle$ for $n \to \infty$, $nL^{-2} = c$.

Proposition

All limit points lead to positive energy representations of $(\overline{\mathfrak{A}}_\alpha, \alpha)$.

(2) (Approximate) ground states and condensates

(a) Ground state: Ω for (renormalized) Hamiltonian $\boldsymbol{H}_r = \boldsymbol{H} + E(N)$.

(b) Approximate (non-Fock) ground states and condensates:

$$\boldsymbol{\Psi}_{L,n} = (n!)^{-1/2}\, \Phi_L \otimes_s \cdots \otimes_s \Phi_L \in \mathcal{F}_n\,, \quad n \in \mathbb{N}$$

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Proposition

All limit points lead to positive energy representations of $(\overline{\mathfrak{A}}_\alpha, \alpha)$.

(3) Particle properties and collision theory

"Particle observables" are uncovered at asymptotic times.

Lemma

Let $V \geq 0$ *be short ranged, and let* $A \in \overline{\mathfrak{A}}$ *be localized. Then (weakly)*

$$\lim_{t\to\infty} \alpha(t)(A) = \langle \Omega, A\Omega \rangle \, 1$$

$$\lim_{t\to\infty} \int d\boldsymbol{x}\, h(\boldsymbol{x}/t)\, \alpha(t,\boldsymbol{x})(A_0) = c_s \int d\boldsymbol{p}\, h(2\boldsymbol{p}) \langle \boldsymbol{p}, A_0\, \boldsymbol{p} \rangle\, \widehat{a}^*(\boldsymbol{p})\widehat{a}(\boldsymbol{p}).$$

Here $A_0 \doteq (A - \langle \Omega, A\Omega \rangle 1)$ *and* $\widehat{\ }$ *indicates "outgoing" operators.*

Similarly for "incoming", collision cross sections etc

Collision theory for observables works [Araki, Haag]

(4) Equilibrium states

Theory defined on $\mathbb{R}^s$ (no boxes). Introduce trapping forces, $L > 0$,

$$\boldsymbol{H}_L \doteq \boldsymbol{H} + \int d\boldsymbol{x}\,(\boldsymbol{x}^2/L^4)\; a^*(\boldsymbol{x})a(\boldsymbol{x})\,.$$

Automorphic action $\alpha_L(t) \doteq \mathrm{Ad}\, e^{itH_L}$ on $\mathcal{B}(\mathcal{F})$, $t \in \mathbb{R}$.

Lemma

$\overline{\mathfrak{A}}$ *stable under action of* $\alpha_L(t)$*, and one has pointwise (in l.c.t.)*

$$lim_{L\to\infty}\alpha_L(t) = \alpha(t) \quad , t \in \mathbb{R}.$$

V of positive type: $\mathrm{Tr}_{\mathcal{F}}\, e^{-\beta(H_L - \mu N)} < \infty$ for $\beta > 0$, $\mu \leq -V(0)$.

$$\omega_{\beta,\mu,L}(\,\cdot\,) = \mathrm{Tr}\,(e^{-\beta(H_L-\mu N)}\,\cdot\,)/\mathrm{Tr}\, e^{-\beta(H_L-\mu N)}$$

KMS-state with regard to $\alpha_L(t)$, $t \in \mathbb{R}$. Limit states exist (Alaoglu).

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Summary

Results:

- resolvent algebra $\mathfrak{R}$ cures "large field problems"
- observable algebra $\overline{\mathfrak{A}}$ composed of AF algebras
- automorphic action of dynamics established for $\overline{\mathfrak{A}}$ and $\overline{\mathfrak{R}}$
- dense C*-dynamical systems exist
- quasi local structure of $\overline{\mathfrak{A}}$ stable under time evolution
- formalism useful for analysis of finite and infinite bosonic systems
- similar results hold for fermionic systems [Bratteli]

Challenges: treatment of

- non-relativistic dynamics changing particle number
- relativistic (canonical) theories in $d = 2$

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Thank you for your attention!