External symmetries of regular maps

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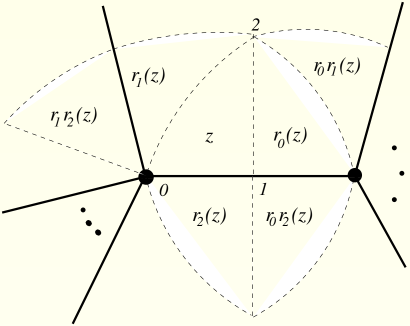

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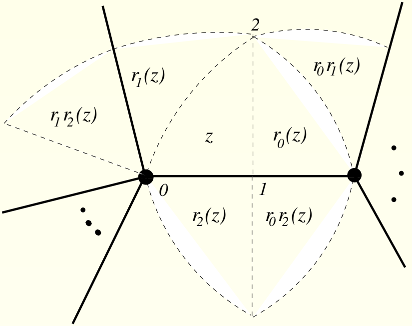

Duality is a result of interchanging the roles of the involutions r_0 and r_2.

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An orientably-regular maps $(H; r, s)$ is positively (negatively) self-dual iff H admits an involutory automorphism interchanging r with s (r with s^{-1}).

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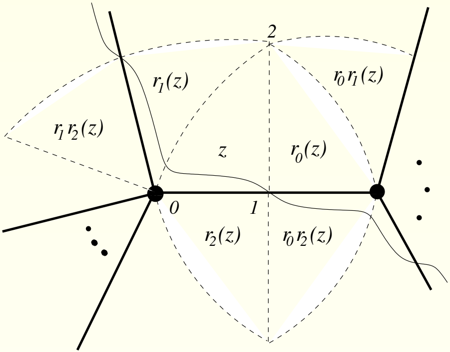
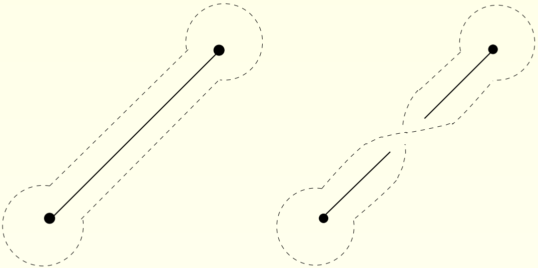


A topological way to form $P(M)$ from M is by introducing a 'half-twist' at every edge of the *ribbon graph*, or the *band-complex* of M:



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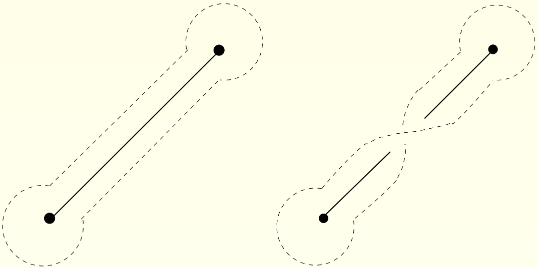

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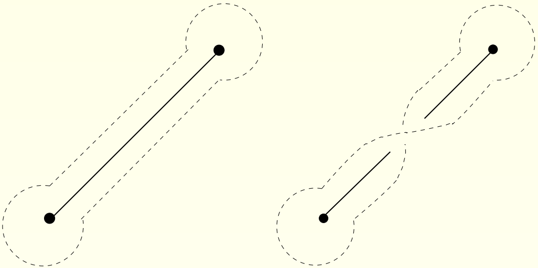

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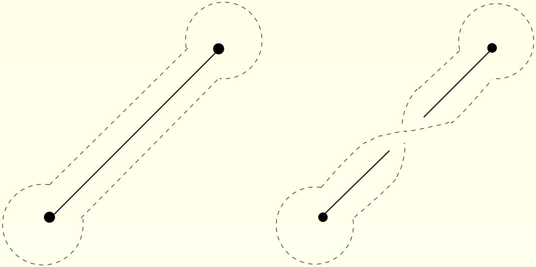

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If M is orientable, then $P(M)$ is orientable iff the graph of M is bipartite.

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Jones and Poulton (2010): For an infinite sequence of valencies m there is a finite orientably regular map of degree m invariant under the operator DP of order 3 but admitting no duality.

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Method: Construction of a suitable U-invariant subspace in D/N, where $D = [T, T]$ for $T = T(m, \infty) = \langle S, T|\ S^m, T^2 \rangle \cong C_m * C_2$, and $N = D' D^{(p)}$.

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Lack of results analogous to the ones for orientably-regular maps ...

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- 3. Structure of the external symmetry group of a kaleidoscopic completely self-dual regular map?
- 4. Is it true that for every $m \geq 3$ and every subgroup U of $C_m^* \times C_2$ there exists a non-orientable regular map of valency m with exponent group U?