Shortest path embeddings of graphs on surfaces

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Joint work with

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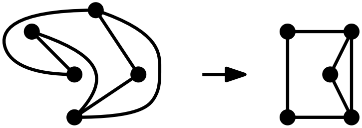
Fáry's theorem

Theorem (Fáry's theorem (Wagner, Fáry, Stein))

Let G be a planar graph. Then G has a plane embedding such that every edge is a straight-line segment.



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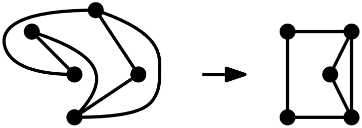
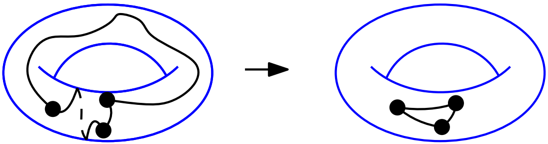


- Is there an analogue on surfaces?



Shortest path embeddings

Definition

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Shortest path embeddings

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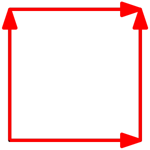
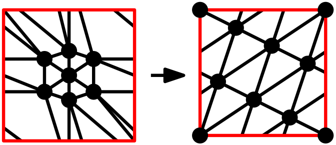
- Let S be a surface equipped with a (Riemannian) metric. An embedding of a graph G into S is a shortest paths embedding if every edge is drawn as the shortest paths between the endpoints.
- A metric on a surface S is a universal shortest path metric if every graph embeddable in S admits a shortest path embedding.
- A metric on S is a k-universal shortest path metric if every graph embeddable in S admits an embedding where each edge is a concatenation of at most k-shortest paths.





The main question

Question

Does there exists a universal shortest paths metric for every surface S? (Or k-universal with a fixed k?)

Motivation:

- Shortest paths embeddings mean a small number of intersections between pairs of graphs embedded in a surface S.
- Negami's conjecture: There is $c > 0$ such that for every G_1, G_2 embedded in S there is a homeomorphism such that $\mathrm{cr}(h(G_1), G_2) \leq c|E(G_1)| \cdot |E(G_2)|$.
- Similar question (for curves on surfaces): Geelen, Huynh, Richter (explicit bounds for graph minors); Matoušek, Sedgwick, T., Wagner (embeddability into 3-space).

Results

Theorem

The sphere, the projective plane, the torus and the Klein bottle can be endowed with a universal shortest paths metric.

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For any $\varepsilon > 0$, with probability tending to 1 as g goes to infinity, a random hyperbolic metric is not a universal shortest paths metric, not even $O(g^{1/3-\varepsilon})$-universal.

Theorem

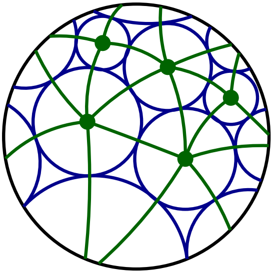
For every $g > 1$, there exists an $O(g)$-universal shortest path hyperbolic metric m on the orientable surface S of genus g.

Sphere and projective plane

Theorem (Stephenson)

Any planar graph can be represented via kissing circles in the sphere.

- Such a representation gives an embedding with shortest paths with respect to the standard round metric on the sphere.
- With uniqueness and symmetry, this also gives that the round metric is shortest path universal on the projective plane.

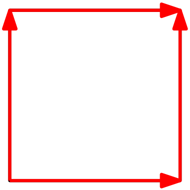


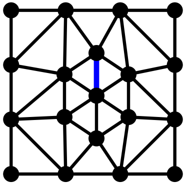
S^2

Minimal triangulations

Definition

A triangulation of a surface is reducible if it contains an edge whose contraction yields again a triangulation. A triangulation is minimal if it is not reducible.





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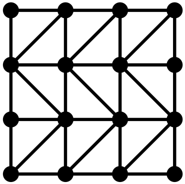
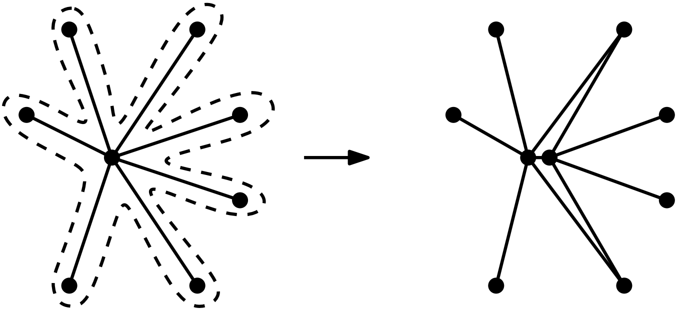


minimal

- For every surface, there is a finite list of minimal triangulations.

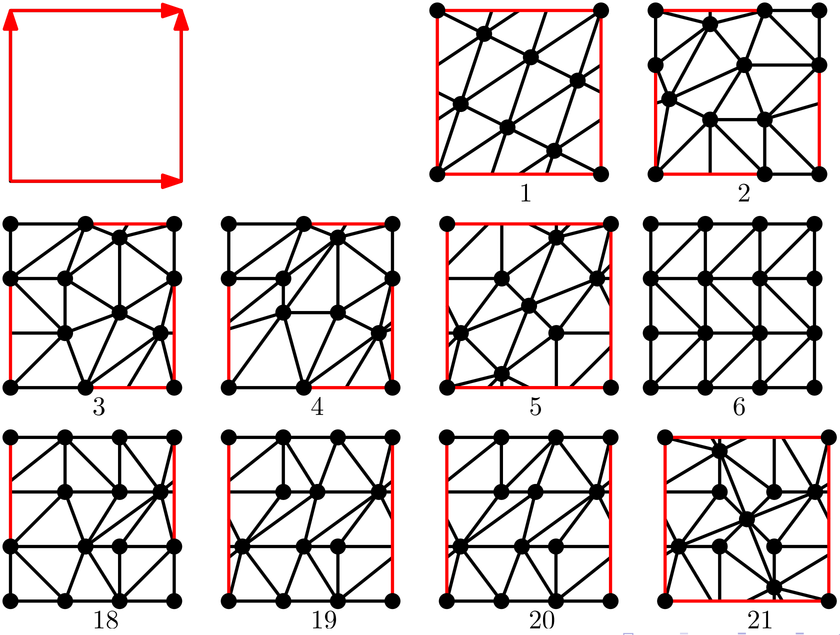
Edge decontractions and shortest paths

- Edge decontractions preserve embeddability with shortest paths.



- For a fixed surface S, it is thus sufficient to check only the minimal triangulations.

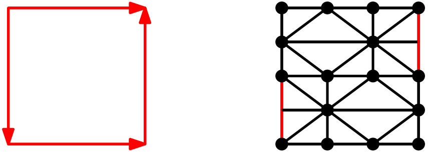
Torus



Klein bottle (square metric is not universal)

Theorem (Answering a question by Schaefer)

The flat square metric on the Klein bottle (w.r.t. polygonal scheme $aba^{-1}b$) is not universal.



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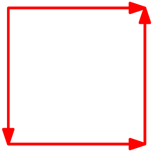

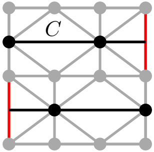

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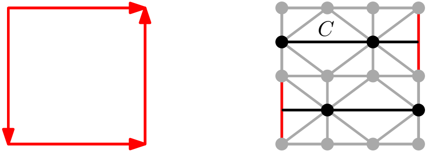

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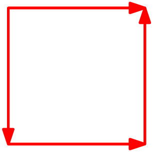

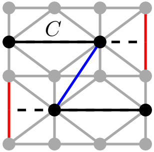

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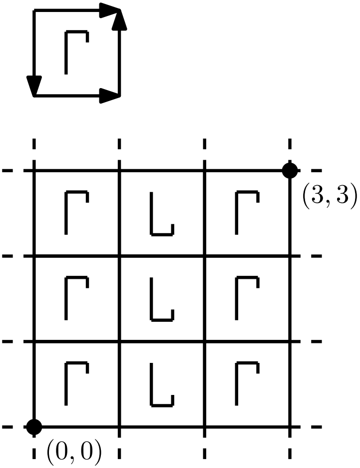
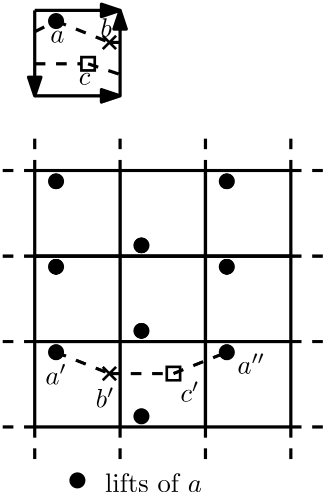
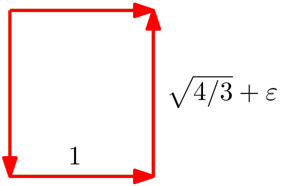
Klein bottle: universal cover





Klein bottle: shortest paths universal metric

- Fix of the problem with the flat square metric:



Random hyperbolic metrics

Theorem

For any $\varepsilon > 0$, with probability tending to 1 as g goes to infinity, a random hyperbolic metric is not a universal shortest paths metric, not even $O(g^{1/3-\varepsilon})$-universal.

- Probabilistic distribution: Weil-Petersson volume (on the moduli space)

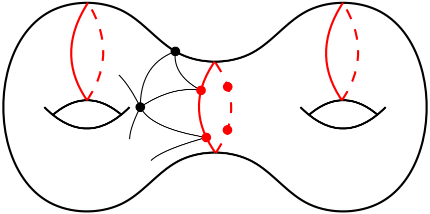
Theorem (Mirzakhani)

The diameter of random hyperbolic surface of genus g is $O(\log g)$ a. a. s.

Theorem (modeled along Guth, Parlier and Young)

For any $\varepsilon > 0$ and any family of types of pants decomposition (ξ_g), a random hyperbolic metric on the surface of genus g has total pants length of type ξ_g at least $\Omega(g^{4/3-\varepsilon})$ a. a. s.

Random hyperbolic metrics



- G with a given pants decomposition.
- Number of edges $O(g)$, total length $\Omega(g^{4/3-\varepsilon})$.
- $\Rightarrow$ There is e of length $\Omega(g^{1/3-\varepsilon})$.
- Endpoints of e in distance at most $O(\log g)$ $\Rightarrow$ e needs $\Omega(g^{1/3-\varepsilon})$ shortest paths.