Packing topological minors half-integrally

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Erdős-Pósa property

A family of graphs $\mathcal{F}$ has the *Erdős-Pósa property* if for every positive integer k, there exists N_k such that every graph either

- contains k disjoint subgraphs each isomorphic to a member of $\mathcal{F}$ or
- contains a set of N_k vertices intersecting every subgraph isomorphic to a member of $\mathcal{F}$.

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Theorem (Erdős,Pósa)

The set of cycles has the Erdős-Pósa property.

Minors

A graph G contains H as a *minor* if H can be obtained from a subgraph of G by contracting edges.

Let $\mathcal{M}(H)$ be the set of graphs containing H as a minor.

Theorem (Robertson, Seymour)

$\mathcal{M}(H)$ has the Erdős-Pósa property if and only if H is planar.

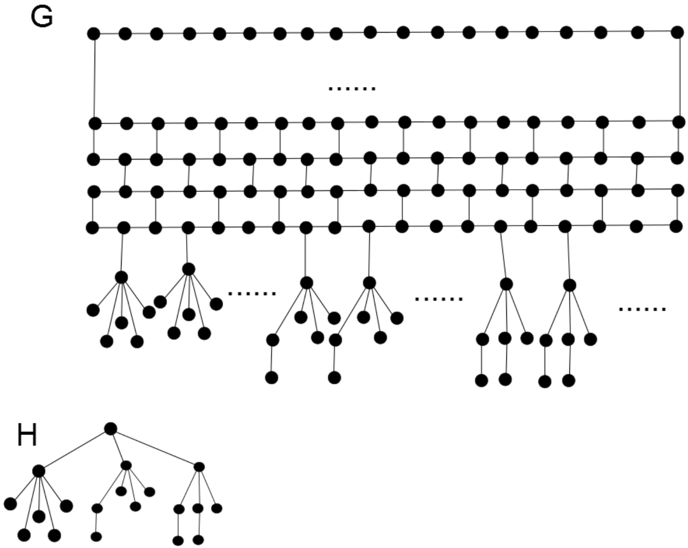
Topological minors

A graph G contains H as a *topological minor* if H can be obtained from a subgraph of G by suppressing vertices of degree two.

Denote the set of graphs containing H as a topological minor by $\mathcal{T}(H)$.

Robertson and Seymour left the following question: for which H does $\mathcal{T}(H)$ have the Erdős-Pósa property?

Some tree has no Erdős-Pósa property



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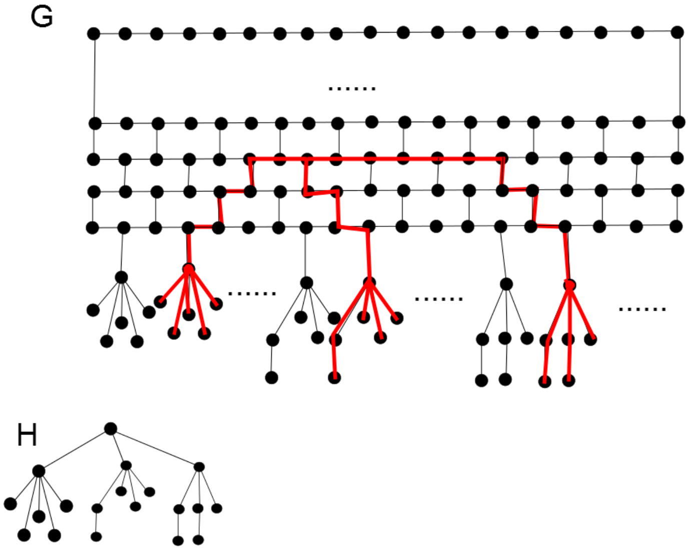
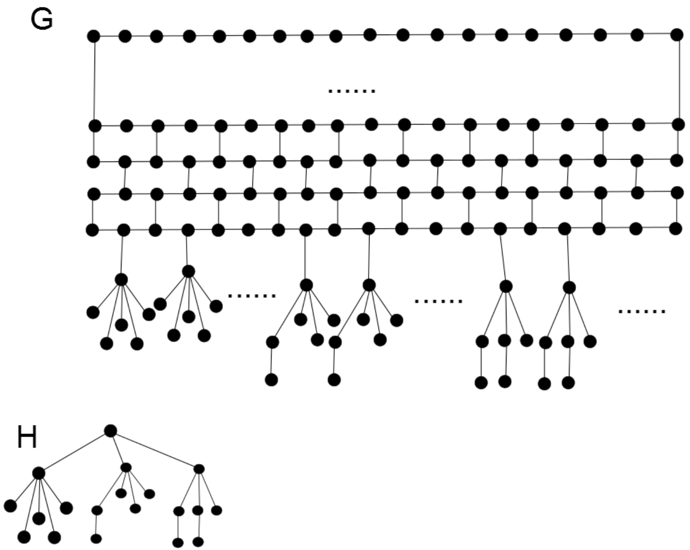


Characterization for topological minors

Theorem (L., Postle, Wollan)

1. *Let H be a graph. Then $\mathcal{T}(H)$ has the Erdős-Pósa property if and only if*
   1. *H can be drawn in the plane such that every vertex of degree at least four is incident with the infinite face.*
   2. *"Maximal" components of H cannot be "covered" by too many "incomparable" subgraphs.*
   3. *Every "covering" of H is "symmetric".*
2. *Deciding whether $\mathcal{T}(H)$ has the Erdős-Pósa property is NP-hard.*

Simpler characterization?



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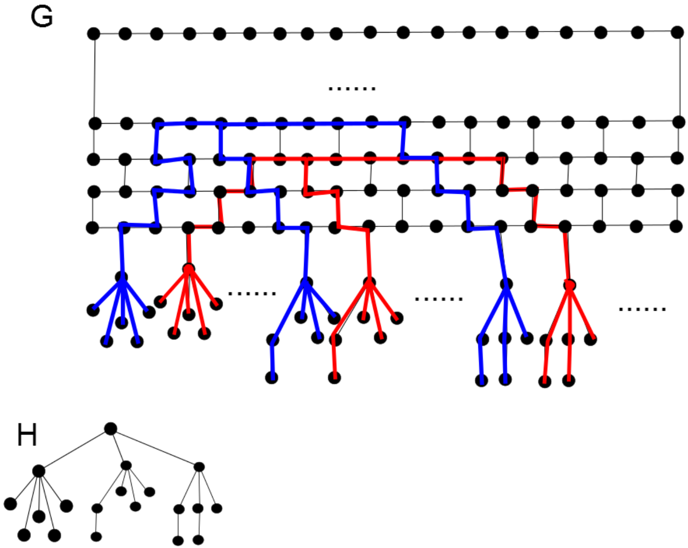

Half-integral packing

A graph G *half-integrally packs* k members of $\mathcal{F}$ if there exist k subgraphs $H_1, H_2, ..., H_k$ of G such that

- each H_i is isomorphic to a member of $\mathcal{F}$, and
- every vertex of G is contained in at most two of $H_1, ..., H_k$.

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A family $\mathcal{F}$ of graphs has the *half-integral Erdős-Pósa property* if for every integer k, there exists N_k such that for every graph G, either

- G half-integrally packs k members of $\mathcal{F}$, or
- there exists $Z \subseteq V(G)$ with $|Z| \leq N_k$ such that Z intersects all subgraphs isomorphic to a member of $\mathcal{F}$.

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Conjecture: (Thomas) For every graph H, $\mathcal{M}(H)$ has the half-integral Erdős-Pósa property.

Half-integrally Erdős-Pósa property

Theorem (Kawarabayashi)

If $H \in \{K_6, K_7\}$, then $\mathcal{M}(H)$ has the half-integral Erdős-Pósa property.

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Theorem (L.)

For every graph H, $\mathcal{T}(H)$ has the half-integral Erdős-Pósa property.

Rooted version

Let $\mathcal{R} = \{R_v : v \in V(H)\}$ be a collection of subsets of $V(G)$.
An *$\mathcal{R}$-rooted subdivision of H* in G is a subdivision of H in G such that for each $v \in V(H)$, its branch vertex corresponding to v belongs to R_v.

Theorem (L.)

For every graph H, there exists a function f such that for every graph G, collection $\mathcal{R}$ and integer k, either

- *G half-integrally packs k $\mathcal{R}$-rooted subdivisions of H, or*
- *there exists $Z \subseteq V(G)$ with $|Z| \leq f(k)$ such that Z intersects all $\mathcal{R}$-rooted subdivisions of H in G.*

Half-integrally packing topological minors

$\mathcal{T}(H) \Rightarrow \mathcal{M}(H)$.

- Let $\mathcal{F}$ be the set of graphs that can be obtained from H by repeatedly splitting vertices of degree at least four.
- A graph contains H as a minor if and only if it contains L as a topological minor for some $L \in \mathcal{F}$.
- For every $L \in \mathcal{F}$, if G half-integrally packs k L-topological minors, then G half-integrally packs k H-minors, so there exists $Z_L \subseteq V(G)$ with $|Z_L| \leq N_{L,k}$ such that $G - Z_L$ has no L-topological minor.
- Let $Z = \bigcup_{L \in \mathcal{F}} Z_L$. Then $G - Z$ has no H-minor.

Applications to structure theorems

Theorem (Dvořák)

For every graph H, there exists r such that every graph G that does not contain H as a topological minor has a tree-decomposition such that every torso either

1. *has at most r vertices of degree at least r, or*
2. *can be "nearly drawn" in a surface in which H cannot be drawn, or*
3. *can be "nearly drawn" in a surface in which H can be drawn in a way that is "nicer" than all possible drawings of H.*

Applications to structure theorems

Theorem (L.)

For every graph H and integer k, there exists r such that every graph G that does not half-integrally pack k subdivisions of H has a tree-decomposition such that every torso either

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Applications to structure theorems

Theorem (L.)

For every graph H and integer k, there exist θ, ξ such that if G is a graph that does not half-integrally pack k subdivisions of H, then for every tangle in G of order at least θ, there exists $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that either

1. *every vertex of $G - Z$ is $(\Delta(H) - 1)$-separable from the tangle, or*
2. *$G - Z$ can be "arranged" in a surface in which H cannot be drawn, or*
3. *$G - Z$ can be "arranged" in a surface in which H can be drawn in a way that is "nicer" than all possible drawings of H.*

Applications to approximation algorithms

A graph H is an *apex-graph* if $H - z$ is planar for some $z \in V(H)$.

Theorem (Demaine, Hajiaghayi)

For every apex-graph H, there exist a polynomial p and constant c such that for any t, there exists an algorithm with approximation ratio $1 - \frac{1}{t}$ and running time $O(c^t p(n))$ for finding maximum size of a stable set on graphs with no H-minor.

The result can be adapted to provides PTASs for general hereditary maximization problems on apex-minor free graphs.

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The result can be adapted to provides PTASs for general hereditary maximization problems on graphs that do not half-integrally pack k apex-minors.

THANK YOU!