Triangle-free graphs with no six-vertex induced path

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BIRS Workshop: Geometric and Structural Graph Theory

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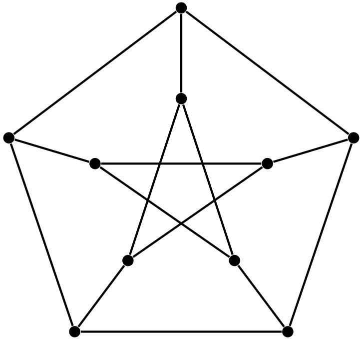
The Clebsch Graph – Construction



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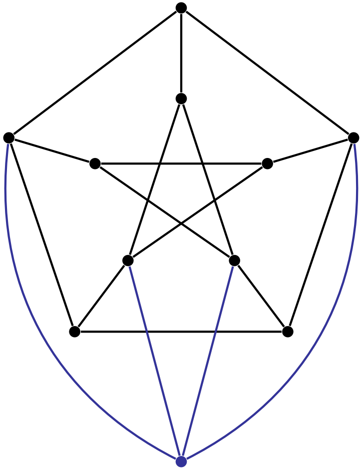

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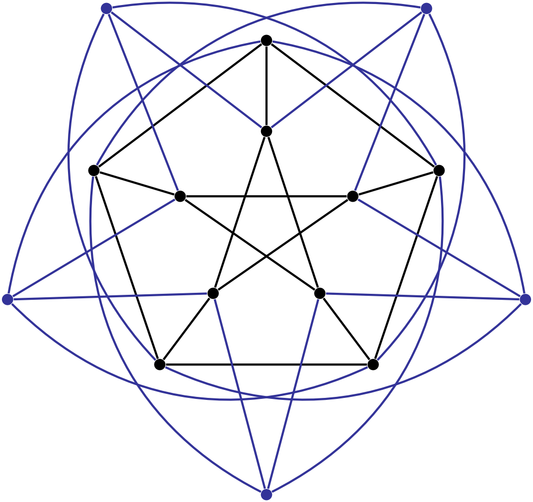

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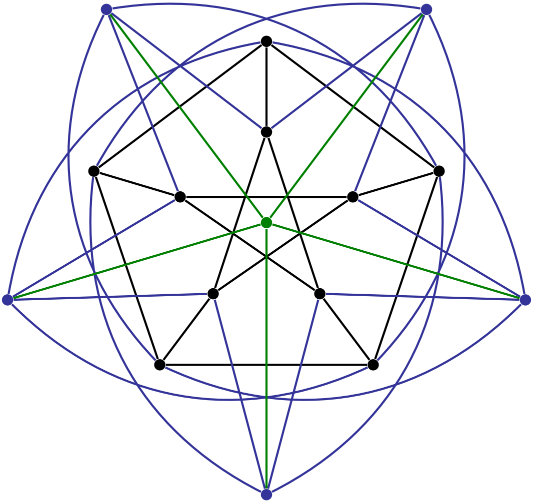
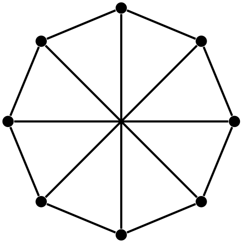


The Clebsch Graph – Another Construction



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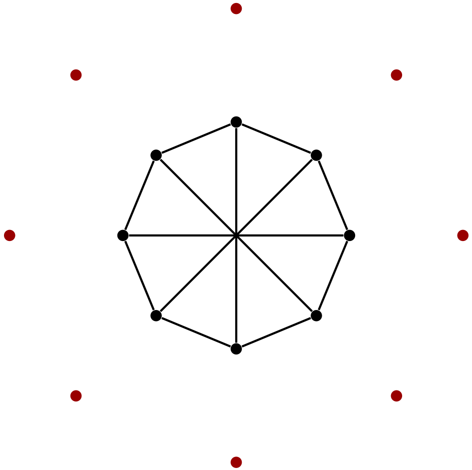

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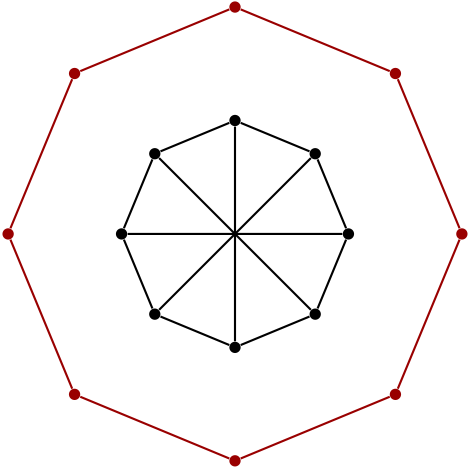

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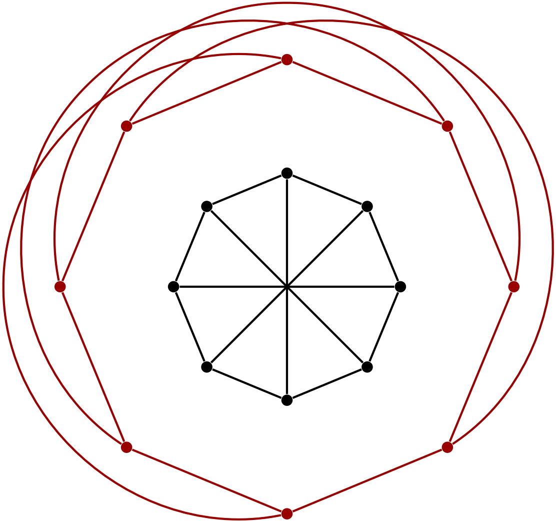

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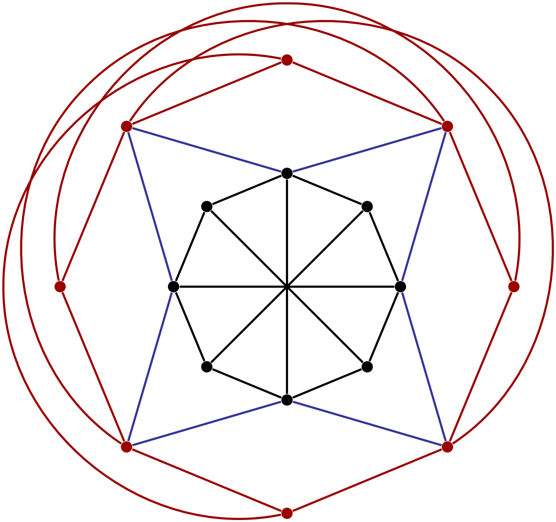

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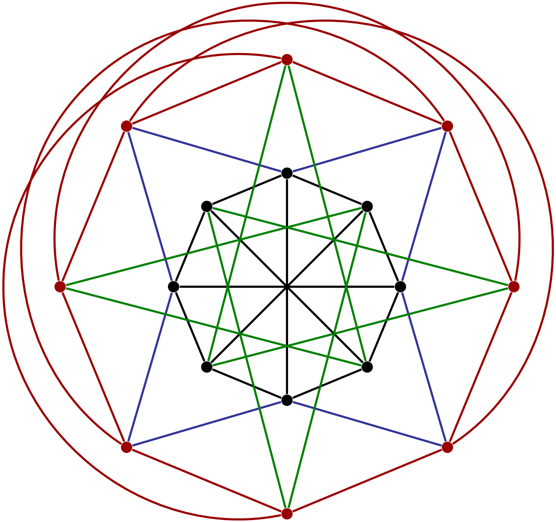
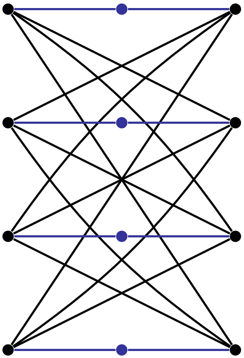


The Clebsch Graph – Properties

- The Clebsch graph is triangle-free, P_6-free, strongly regular, and vertex-transitive
- The edges of K_{16} can be partitioned into three copies of the Clebsch graph (thus $R(3,3,3) \geq 17$)[1]
- May identify its vertices with elements of $GF(16)$ such that two vertices are adjacent if and only if the difference between the corresponding elements is a cube
- Removing any vertex and its neighbors yields the Petersen graph

[1] Greenwood, Gleason (1955)

Another $\{P_6, \text{triangle}\}$-Free Graph



Take $K_{n,n}$, subdivide a perfect matching

- A graph is **scalable** if it is an induced subgraph of this for some n.

Main Result

Theorem (Chudnovsky, Seymour, S., Zhong)

Let G be a connected $\{P_6, \text{triangle}\}$-free graph without twins. Then either

- *G admits a nontrivial simplicial homogeneous pair;*
- *G is a V_8-expansion;*
- *G is an induced subgraph of the Clebsch graph; or*
- *G is scalable.*

Previous results:

- **Randerath, Schiermeyer, Tewes:** Let G be a connected $\{P_6, \text{triangle}\}$-free graph in which no two vertices dominate each other. Then either G is 3-colorable, or G is an induced subgraph of the Clebsch graph.
- **Brandstädt, Klembt, Mahfud:** $\{P_6, \text{triangle}\}$-free graphs have bounded clique-width.

Main Result

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Nontrivial simplicial homogeneous pair: $A, B \subseteq V(G)$, stable, disjoint, with $|A| + |B| \geq 3$, $A \cup B$ not stable, such that no vertex of $V(G)$ has any of the following:

- a neighbor and a non-neighbor in A;
- a neighbor and a non-neighbor in B;
- a neighbor in A and a neighbor in B.

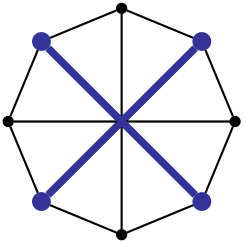
Main Result

Theorem (Chudnovsky, Seymour, S., Zhong)

Let G be a connected $\{P_6, \text{triangle}\}$-free graph without twins. Then either

- *G admits a nontrivial simplicial homogeneous pair;*
- *G is a V_8-expansion;*
- *G is an induced subgraph of the Clebsch graph; or*
- *G is scalable.*

V_8-expansion: Each blue edge is replaced by an **antisubmatching**, a bipartite graph such that every vertex has at most one non-neighbor on the opposite side. May delete black vertices.



$(K_2 + P_3)$-Free Graphs

Theorem

Let G be a connected $\{P_6, \text{triangle}\}$-free graph without twins that contains $K_2 + P_3$. Then either

- *G admits a nontrivial simplicial homogeneous pair; or*
- *G is a V_8-expansion.*

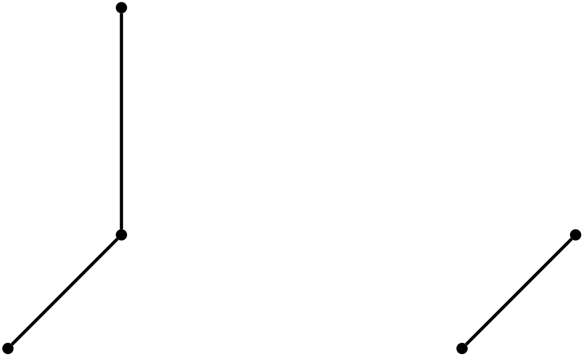
Theorem

Let G be connected and $\{K_2 + P_3, \text{triangle}\}$-free. Then either:

- *G admits a nontrivial submatched simplicial homogeneous pair;*
- *G may be obtained from an induced subgraph of the Clebsch graph by safely adding twins;*
- *G is scalable, or an extended antisubmatching; or*
- *G is a half-graph expansion.*

$\{P_6, \text{triangle}\}$-Free Graphs Containing $K_2 + P_3$

Let G connected, $\{P_6, \text{triangle}\}$-free, containing $K_2 + P_3$, with no twins.



- If $K_2 + P_3$ is present, we may assume that this graph is present

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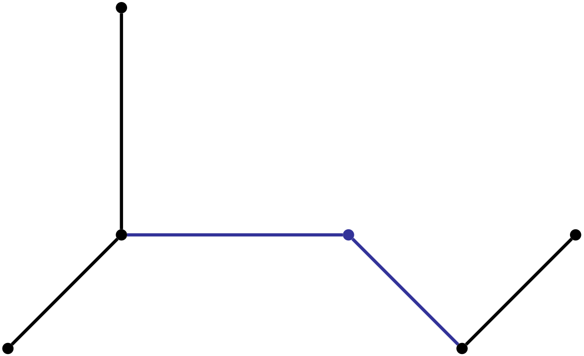

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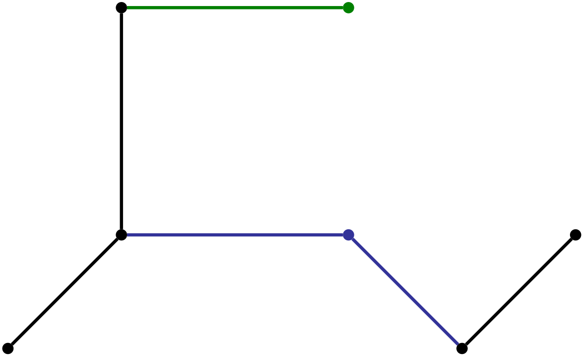

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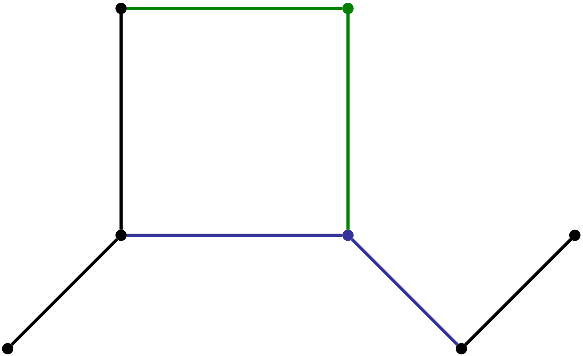

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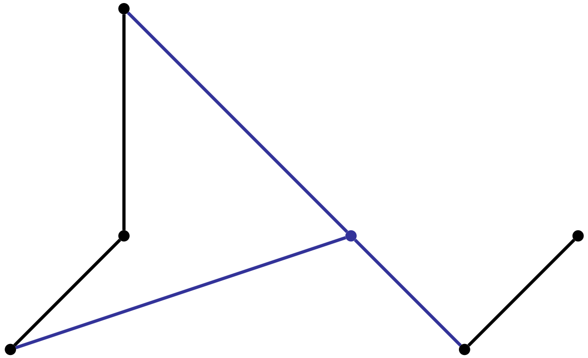

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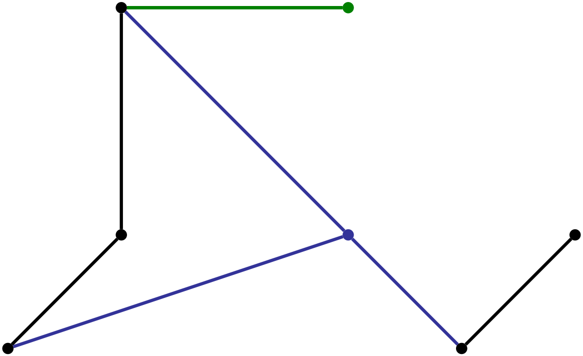
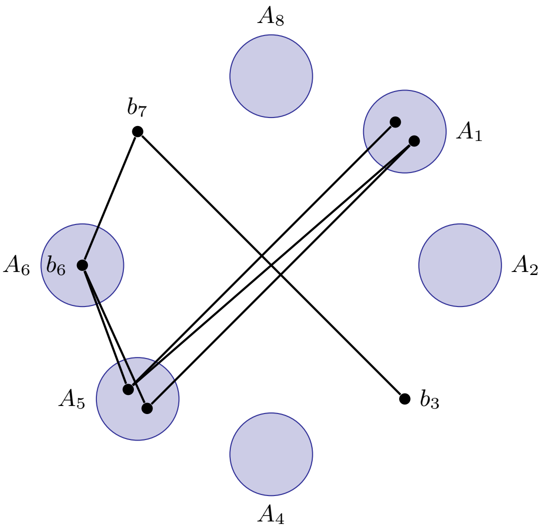


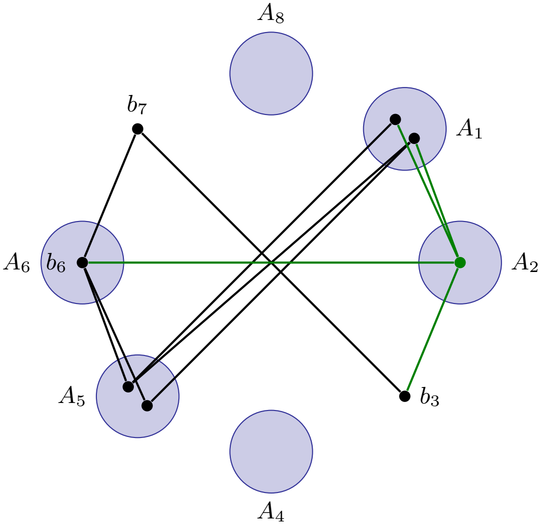
- If $K_2 + P_3$ is present, we may assume that this graph is present

V_8-Expansion



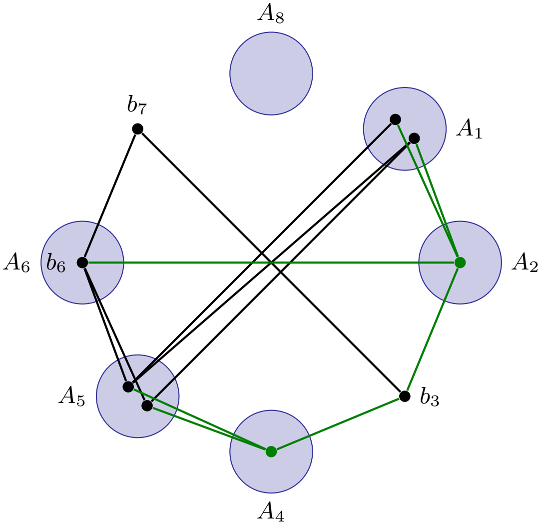
- A_1, A_5 maximal with $G[A_1 \cup A_5]$ connected, A_5 complete to b_6, no other edges

V_8-Expansion



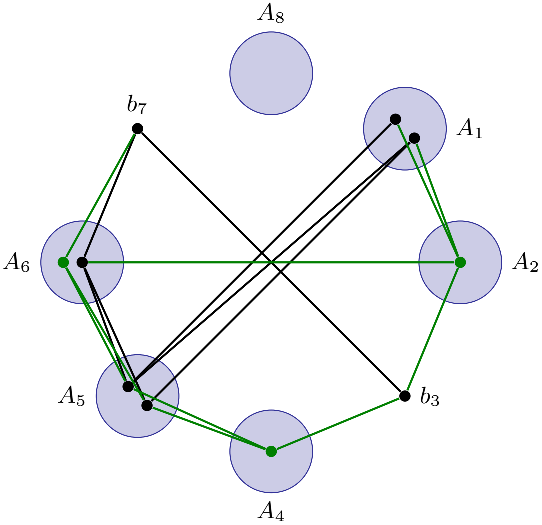
- Let A_2 be the set of vertices complete to $A_1 \cup \{b_3, b_6\}$

V_8-Expansion



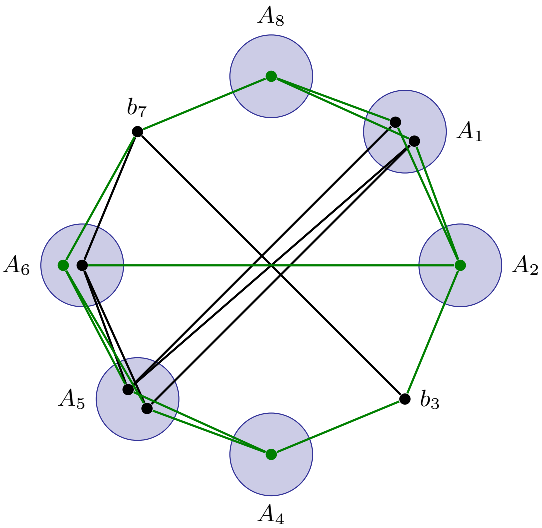
- Let A_4 be the set of vertices complete to $A_5 \cup \{b_3\}$

V_8-Expansion



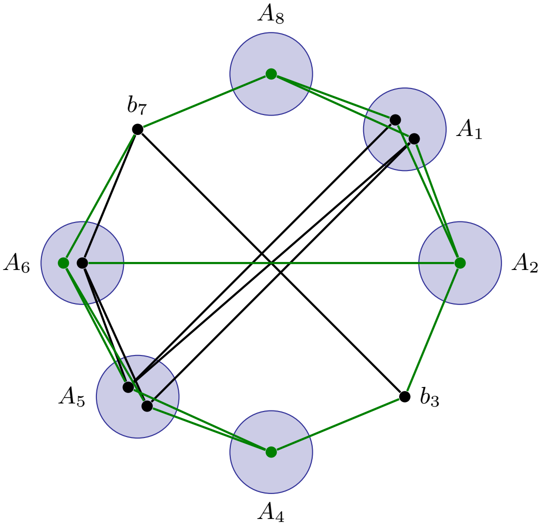
- Let A_6 be the set of vertices complete to $A_5 \cup \{b_7\}$

V_8-Expansion



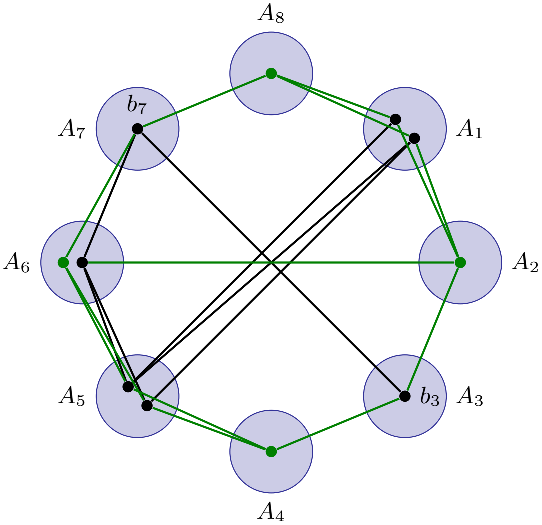
- Let A_8 be the set of vertices complete to $A_1 \cup \{b_7\}$

V_8-Expansion



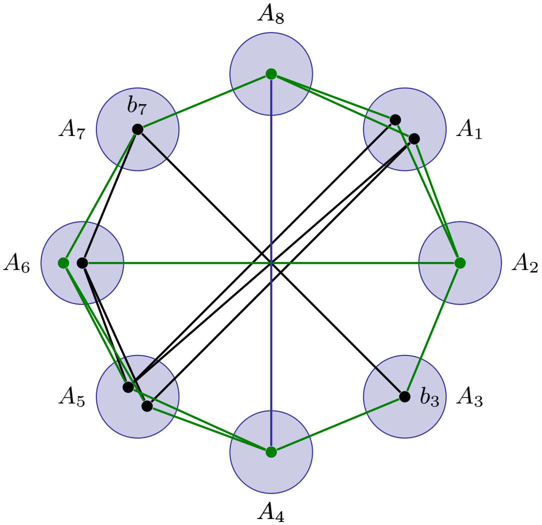
- Every vertex with a neighbor in $A_1 \cup A_5$ is in one of these sets

V_8-Expansion



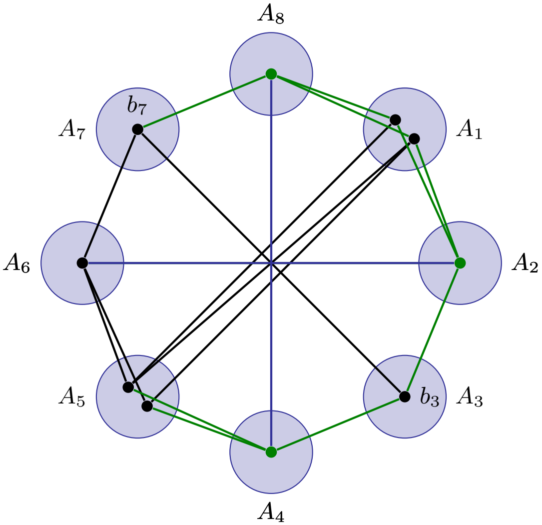
- A_3, A_7 maximal with $b_3 \in A_3, b_7 \in A_7$, $G[A_3 \cup A_7]$ connected, A_7 complete to b_6

V_8-Expansion



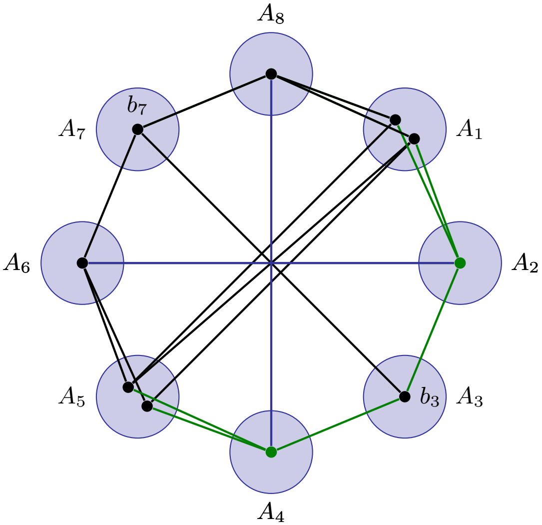
- A_4 is complete to A_8

V_8-Expansion



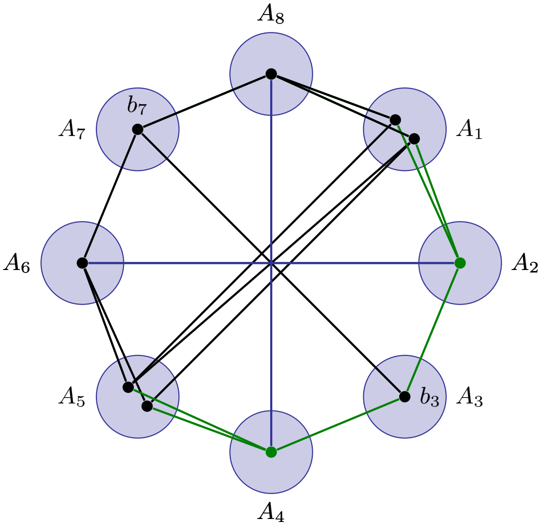
- A_4 is complete to A_8, and A_2 is complete to A_6

V_8-Expansion



- If the neighbors of A_1 are complete to those of A_5, win; so WMA $A_8 \neq \emptyset$

V_8-Expansion



- Every remaining vertex with a neighbor in $A_6 \cup A_8$ is complete to $A_6 \cup A_8$

Thank you!