Coloring Graphs with Forbidden Minors

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Brief Background

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- If $\chi(G) = t$, then G contains a K_t minor ???

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- Not even known yet whether every graph with no K_7 minor is 7-colorable.

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- **G has no $K_p^=$ minor** $\iff$ G contains neither of the two graphs in $K_p^=$ as a minor

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- $t = 7$:
  - If $G \not> K_7^{=}$, then G is 6-colorable. Jakobsen (1971)
  - If $G \not> K_7^{-}$, then G is 7-colorable. Jakobsen (1983)
  - If $G \not> K_7$, then G is 9-colorable. Mader (1968)
  - If $G \not> K_7$, then G is 6-colorable or $G > K_{4,4}$. Kawarabayashi & Toft (2005)
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- $t = 9$:
  - If $G \not> K_9$, then G is 13-colorable. S. & Thomas (2006)

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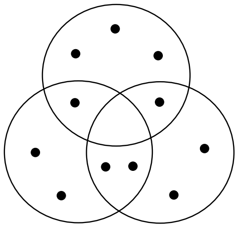
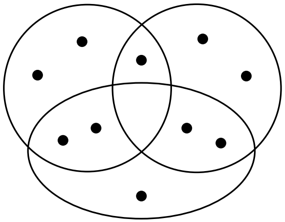
LEM (Rolek & S. 2015++): If $G \neq K_8$ is an 8-contraction-critical graph having two different K_5's with exactly three vertices in common or three different K_5's as depicted below, then $G > K_7$.



(a)



(b)

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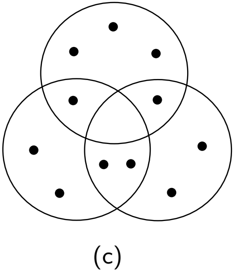

- Applied rooted K_4-minor result to the case when two K_5's have exactly three vertices in common.

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- $n_8 \leq 2$.
- for any $v \in V(G)$ with $d_G(v) = 9$, either $G[N(v)]$ contains K_4 as a subgraph or $0 \leq \delta(G[N(v)]) \leq 4$.

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- $8 \leq \delta(G) \leq 9$ and $2n_8 + n_9 \geq 30$.
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- We were unable to prove that G is 8-connected.

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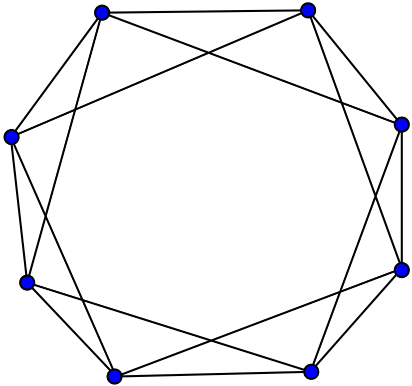

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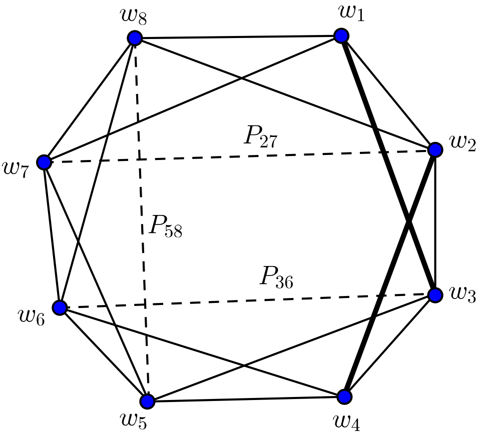

LEM (Rolek & S. 2015++): Let H be a graph with $|H| = 9$ and $\delta(H) \geq 5$. Then either $H > K_6$, or H is isomorphic to one of the 17 graphs. Moreover, if H is K_4-free, then either $H > K_6$, or H is isomorphic to $\overline{K_3} + C_6$.

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An edge e is a **missing edge** in G if $e \in E(\overline{G})$.

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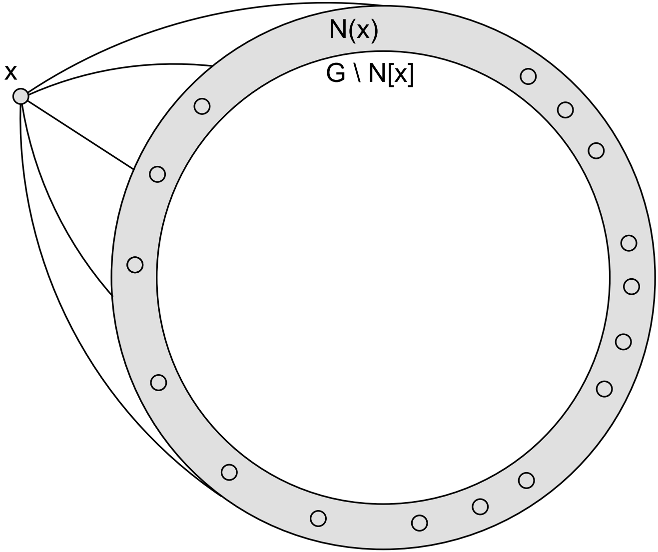
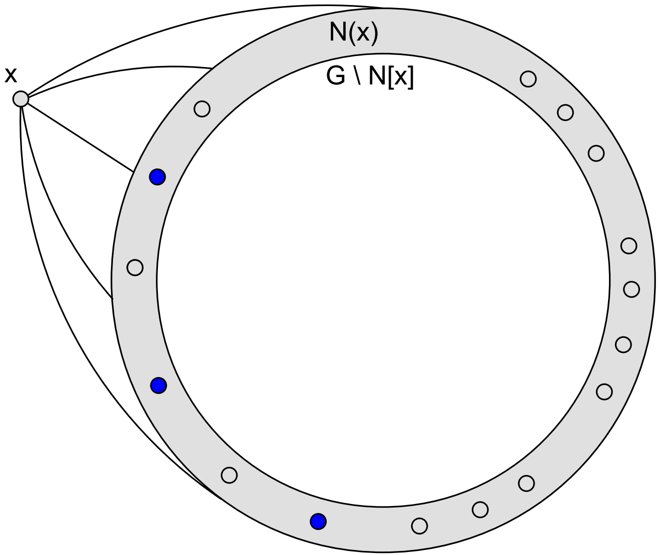
LEM (Rolek & S. 2017): Let G be any k-contraction-critical graph. Let $x \in V(G)$ be a vertex of degree $k+s$ with $\alpha(G[N(x)]) = s+2$ and let $S \subset N(x)$ with $|S| = s+2$ be any independent set, where $k \geq 4$ and $s \geq 0$ are integers. Let M be a set of missing edges of $G[N(x)\backslash S]$. Then there exists a collection $\{P_{uv} : uv \in M\}$ of paths in G such that for each $uv \in M$, P_{uv} has ends $\{u, v\}$ and all its internal vertices in $G\backslash N[x]$. Moreover, if vertices u, v, w, z with $uv, wz \in M$ are distinct, then the paths P_{uv} and P_{wz} are vertex-disjoint.


N(x)
x
G \ N[x]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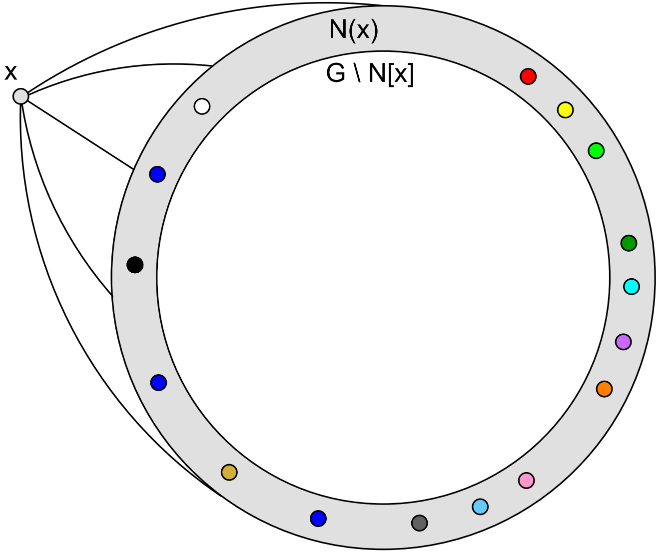
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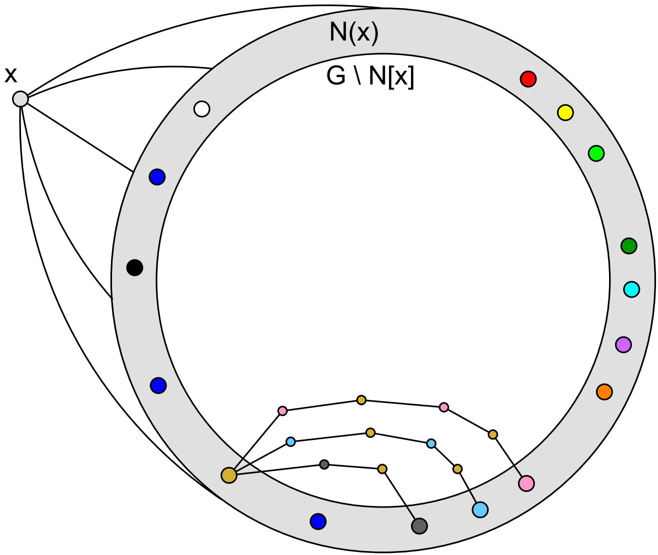
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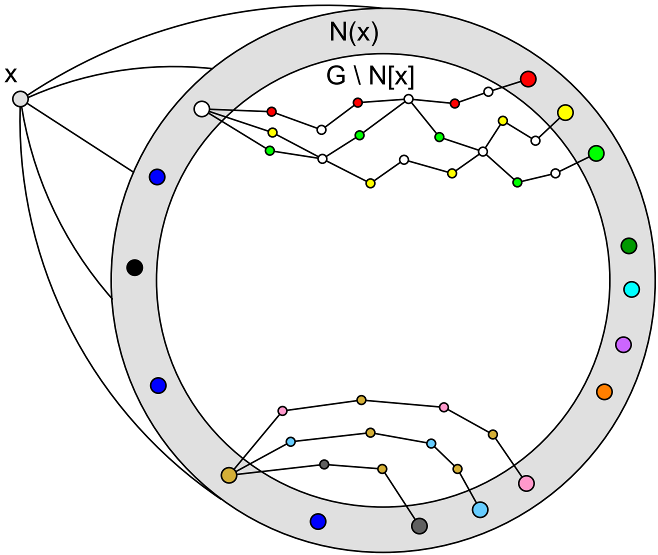
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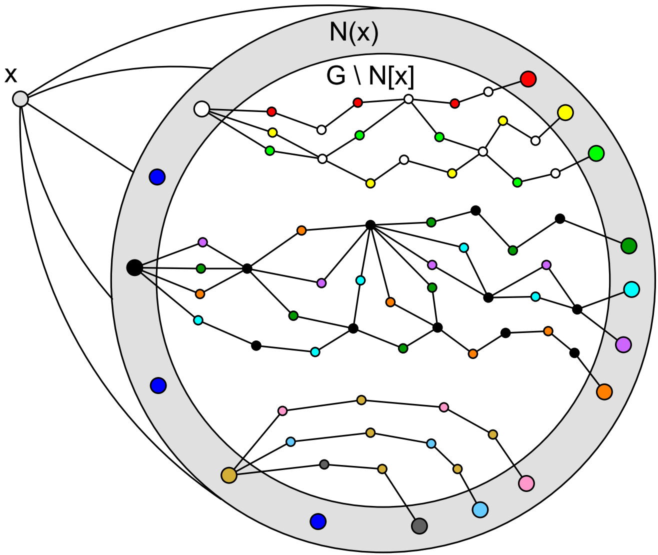

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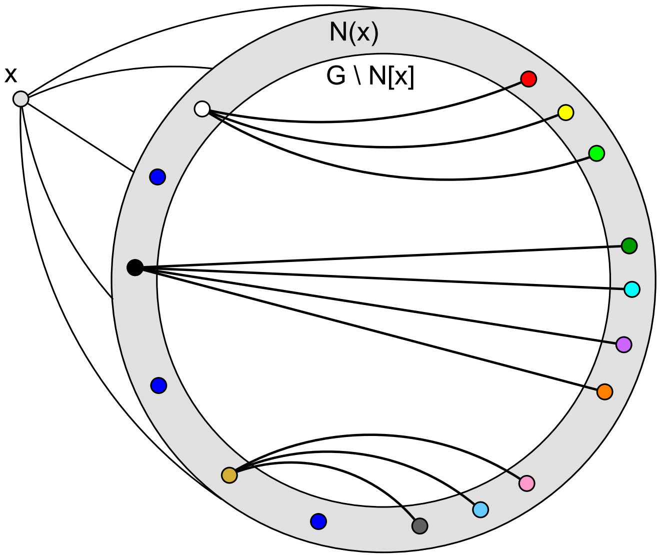
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- **Our proofs for $t = 7, 8$ are short and computer-free.**

An Application of Wonderful Lemma

Conjecture (Rolek & S. 2017): For every $t \geq 1$, every graph G on $n \geq t$ vertices and at least $(t-2)n - \binom{t-1}{2} + 1$ edges either contains a K_t minor or is $(t-1)$-colorable.

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Proof Sketch

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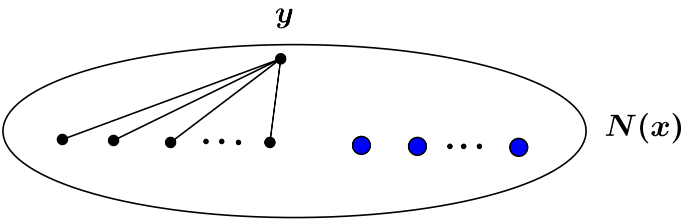

$$d_{N(x)}(y) = t - 3$$

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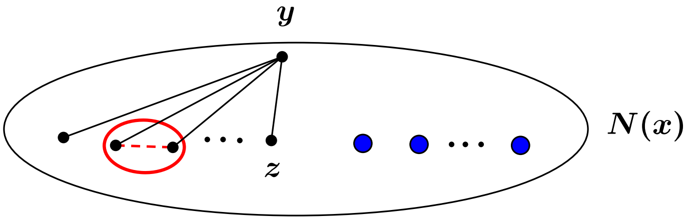
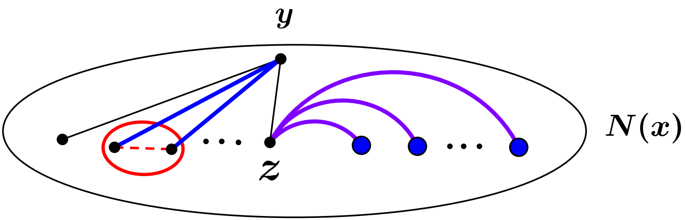


$$d_{N(x)}(y) = t - 3 \geq 3$$

- $t \geq 6$ is only required here.

Proof Sketch

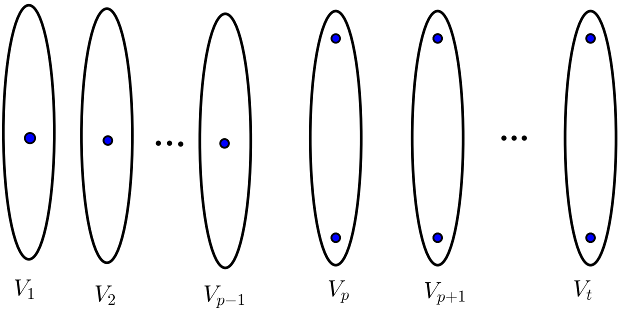


$$d_{N(x)}(y) = t - 3 \geq 3$$

- y has $t-3$ neighbors and $t-3$ non-neighbors.
- Contracting the blue seagull into a single vertex, all purple paths onto z yield a K_{t-1} minor in $N(x)$.

Proof Sketch

Case 1: $\chi(G[N(x)]) \geq t$



where $V_1, V_2, \ldots, V_t$ are the color classes of $G[N(x)]$.

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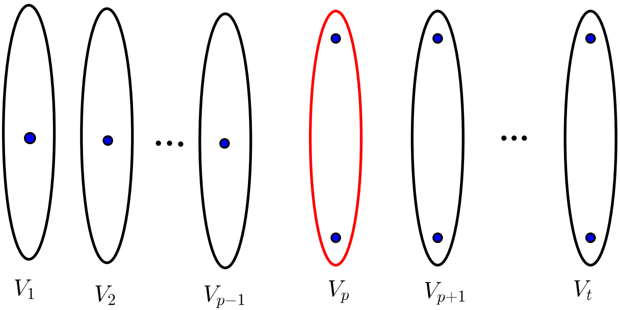

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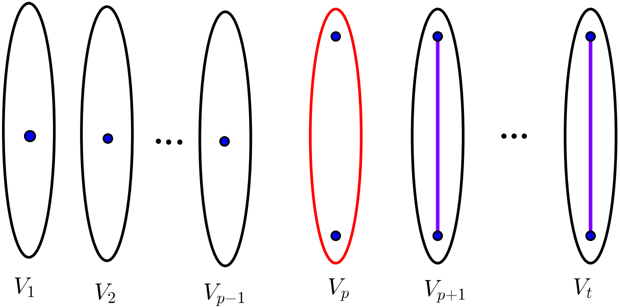

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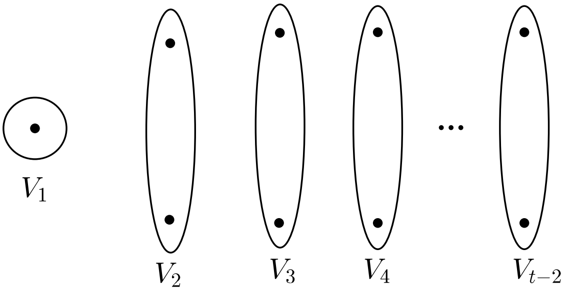
Proof Sketch

Case 2: $\chi(G[N(x)]) = t - 2$



- exact one singleton.

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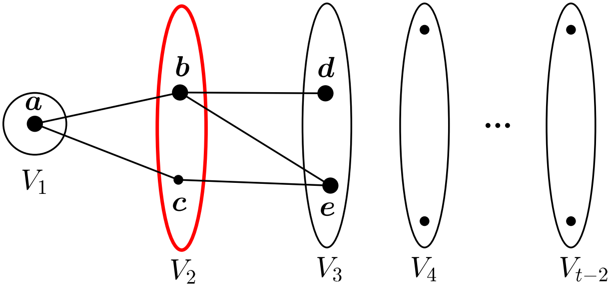

- a must be complete to some V_i, say V_2.
- b, c must have a common neighbor in some V_j, say $e \in V_3$.
- We may assume that $db \in E(G)$.

Proof Sketch

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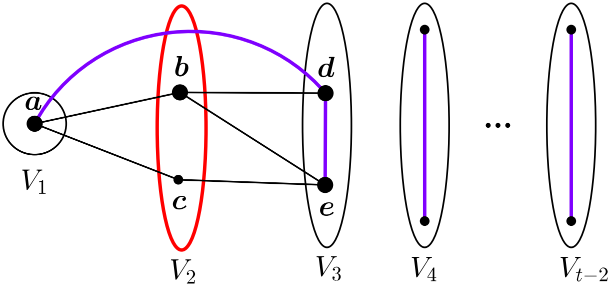
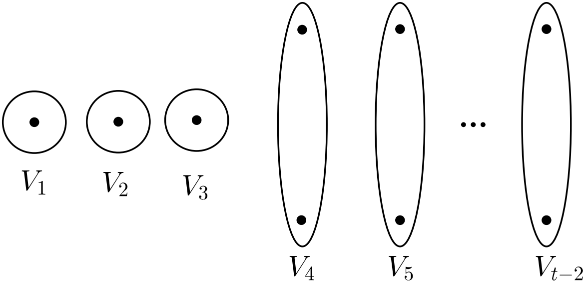


- P: ad-path; Q: ed-path.
- Contracting $P - a$ and $Q - e$ onto d.
- Contracting the edge ce.

Proof Sketch

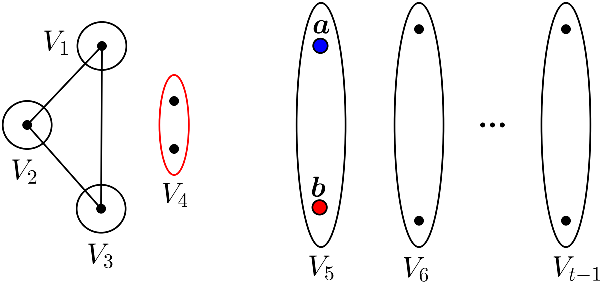
Case 3: $\chi(G[N(x)]) = t - 1$



- exact three singletons.

Proof Sketch

Case 3: $\chi(G[N(x)]) = t - 1$



- Each vertex in $V_1 \cup V_2 \cup V_3$ is adjacent to either a or b.
- Assume a has more neighbors in $V_1 \cup V_2 \cup V_3$ than b.

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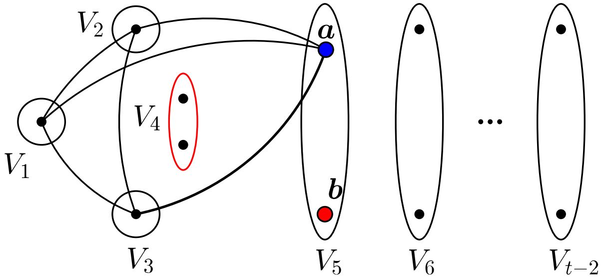

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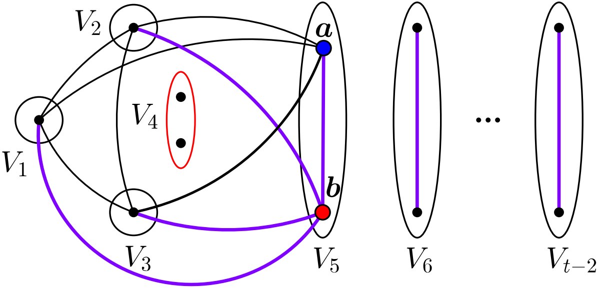

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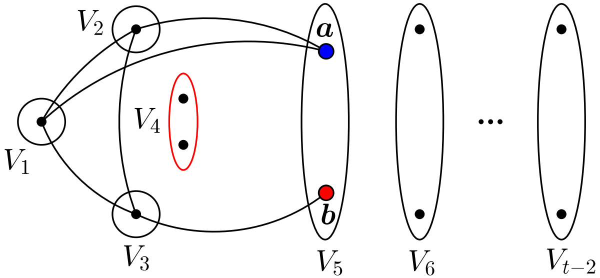

- a is complete to $V_1 \cup V_2$ and b is complete to V_3.

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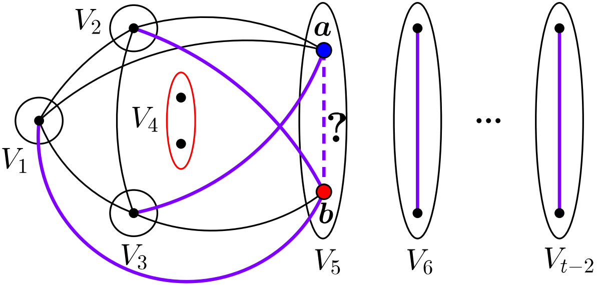

- a is adjacent to exactly two of the three singletons.
- aV_3-path is disjoint from bV_1- and bV_2-path.
- ab-path may intersect with each of aV_3-, bV_1- and bV_2-path.

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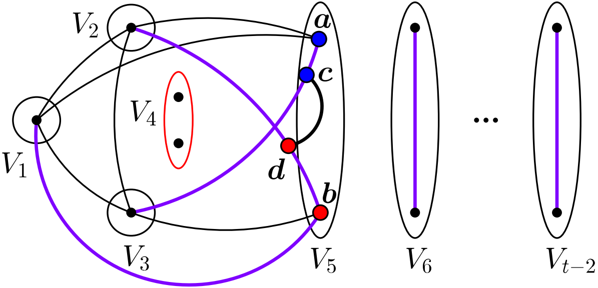

- d: first vertex on the ab-path (when read from a to b) which is also on the bV_1 or bV_2-path.
- c: first vertex on the aV_3-path (when read from V_3 to a) which is also on the da-subpath of the ba-path.
- $d \neq a$, $c \neq b$, cd-subpath is disjoint from cV_3-subpath.

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  Remark: Seymour-Thomas Conjecture is open for $p \geq 10$.

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- **THM**(Jakobsen 1972): Every graph on $n \geq 8$ vertices and at least $5n - 14$ edges either has a $K_8^=$ minor or is a $(K_7, 4)$-cockade.

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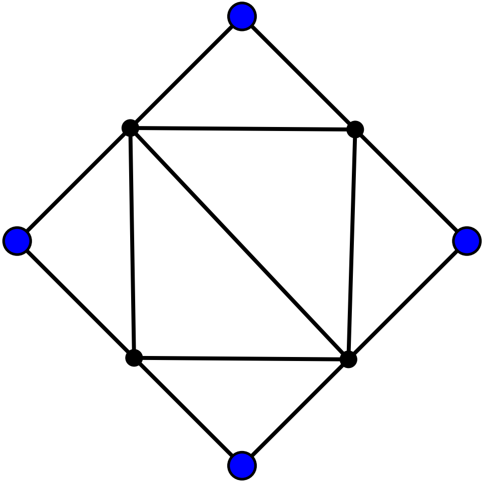

- square of this chordal graph is not chordal.

THANK YOU!