Coloring a class of $4K_1$-free graphs

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Coloring $\mathcal{H}$-free graphs

Question

What is the complexity of coloring $\mathcal{H}$-free graphs?

Where $\mathcal{H}$ is any family of graphs.

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Where $\mathcal{H}$ is any finite (small) family of (small) graphs.

One forbidden subgraph

When $\mathcal{H}$ has only one member:

Theorem (Král', Kratochvil, Tuza, Woeginger 2001)

Coloring H-free graphs is:

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Coloring H-free graphs is:

- *Polynomially solvable when H is an induced subgraph of either P_4 or $P_3 + P_1$.*

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Coloring H-free graphs is:

- *Polynomially solvable when H is an induced subgraph of either P_4 or $P_3 + P_1$.*
- *NP-complete in all other cases.*

Two forbidden subgraphs

When $\mathcal{H}$ has two members H_1, H_2:

Theorem (Golovach, Johnson, Paulusma, Song 2016)

Coloring (H_1, H_2)-free graphs is polynomially solvable when:

Two forbidden subgraphs

When $\mathcal{H}$ has two members H_1, H_2:

Theorem (Golovach, Johnson, Paulusma, Song 2016)

Coloring (H_1, H_2)-free graphs is polynomially solvable when:

1. H_1 or H_2 is an induced subgraph of P_4 or $P_3 + P_1$.
2. $H_1 \leq K_{1,3}$, and $H_2 \leq$ either bull, hammer, or P_5.
3. $H_1 \leq$ paw, and $H_2 = K_{1,3} + 3P_1$ or H_2 is a forest on at most 6 vertices $\neq K_{1,5}$.
4. $H_1 = K_t$ for $t \geq 4$, and $H_2 \leq$ either sP_2 or $sP_1 + P_5$ $(t, s$ fixed$)$.
5. $H_1 \leq$ paw, and $H_2 \leq$ either sP_2 or $sP_1 + P_5$ $(s$ fixed$)$.
6. $H_1 \leq$ gem, and $H_2 \leq$ either $P_1 + P_4$ or P_5.
7. $H_1 \leq$ house, and $H_2 \leq$ either $P_1 + P_4$ or P_5.
8. $H_1 \leq 2P_1 + P_2$, and $H_2 \leq$ either 4-wheel, $\overline{2P_1 + P_3}$, $\overline{P_2 + P_3}$.
9. $H_1 \leq$ diamond, and $H_2 \leq$ either $P_1 + 2P_2$ or $2P_1 + P_3$ or $P_2 + P_3$.
10. $H_1 \leq tP_1 + P_2$, and $H_2 \leq$ either P_5 or $sP_1 + P_2$ $(t, s$ fixed$)$.
11. $H_1 \leq 4P_1$, and $H_2 \leq \overline{2P_1 + P_3}$.
12. $H_1 \leq P_5$, and $H_2 \leq$ either C_4 or $\overline{2P_1 + P_3}$.

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Theorem (Golovach, Johnson, Paulusma, Song 2016)

Coloring (H_1, H_2)-free graphs is NP-complete when:

1. $H_1 \geq C_r$ $(r \geq 3)$ and $H_2 \geq C_s$ $(s \geq 3)$.
2. $H_1 \geq$ claw, and $H_2 \geq$ either claw, or $\overline{2P_1 + P_2}$ or C_r $(r \geq 4)$ or K_4 or $\Phi_{i,j}$ $(i, j$ even$)$ or Φ'_i $(i$ odd$)$ or Φ''_i $(i$ even$)$.
3. $H_1 \geq \overline{\Phi_i}$ $(i \geq 1)$, and $H_2 \geq$ any 4-vertex subgraph of $2P_2$.
4. H_1 and $H_2 \geq$ any 4-vertex subgraph of $2P_2$.
5. $H_1 \geq$ bull, and $H_2 \geq$ either $K_{1,4}$ or $\overline{C_4 + P_1}$.
6. $H_1 \geq C_3$ and $H_2 \geq K_{1,r}$, $r \geq 5$.
7. $H_1 \geq C_3$ and $H_2 \geq P_{22}$.
8. $H_1 \geq C_r$ $(r \geq 5)$, and $H_2 \geq$ any 4-vertex subgraph of $2P_2$.
9. $H_1 \geq C_3 + P_1$ or $C_4 + P_1$ or $\overline{C_r}$ $(r \geq 6)$, and $H_2 \geq$ any 4-vertex subgraph of $2P_2$.
10. $H_1 \geq K_5$ and $H_2 \geq P_7$.
11. $H_1 \geq K_6$ and $H_2 \geq P_6$.

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8. $H_1 \geq C_r$ $(r \geq 5)$, and $H_2 \geq$ any 4-vertex subgraph of $2P_2$.
9. $H_1 \geq C_3 + P_1$ or $C_4 + P_1$ or $\overline{C_r}$ $(r \geq 6)$, and $H_2 \geq$ any 4-vertex subgraph of $2P_2$.
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Many open cases remain.

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Lozin and Malyshev proved that the last two cases are polynomially equivalent.

Our contribution

Theorem

There is a polynomial-time algorithm for coloring (claw, $4K_1$, $K_5 \setminus e$)-free graphs.

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- We may assume that G is connected and contains a stable set of size 3.
  (Otherwise, coloring reduces to matching in $\overline{G}$.)

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Theorem

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Sketch of proof:

- We may assume that G is connected and contains a stable set of size 3.
  (Otherwise, coloring reduces to matching in $\overline{G}$.)
- We may assume that G is not perfect.
  (Otherwise used Hsu 1981 or M. and Reed 1999.)

Therefore G is connected, not perfect and contains a stable set of size 3.

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Lemma (Ben Rebea, see Chvátal and Sbihi 1988)

Let G be a connected claw-free graph that contains a stable set of size 3. If G contains an odd antihole, then G contains a 5-hole.

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Since there is no stable set of size 4, G contains a 5-hole or 7-hole H.

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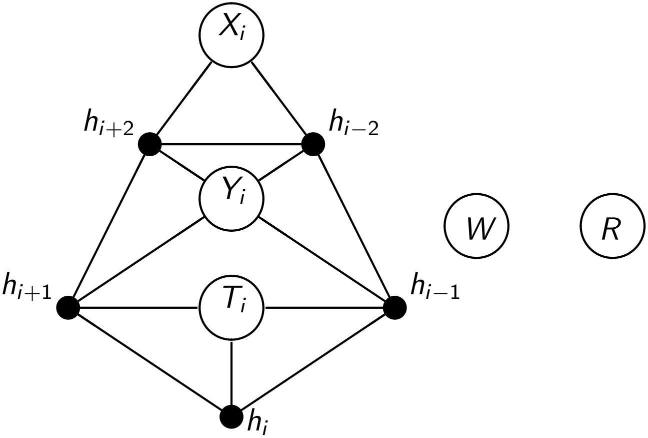
Lemma

If G contains a 7-hole, then $|V(G)| \leq 28$.

When G contains a 5-hole



W = vertices that are complete to the 5-hole.

R = vertices that are anticomplete to the 5-hole.

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4. R *is a clique, and* R *is complete to* X *and anticomplete to* $T \cup Y \cup W$.
5. *If* $R \neq \emptyset$, *then either* $|V(G)| \leq 24$ *or* G *has a clique cutset.*

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4. R *is a clique, and* R *is complete to* X *and anticomplete to* $T \cup Y \cup W$.
5. *If* $R \neq \emptyset$*, then either* $|V(G)| \leq 24$ *or* G *has a clique cutset.*
6. *If* X_i *is "large", then either* X_{i-1} *and* X_{i+1} *are both "small", or one of* X_{i-1} *and* X_{i+1} *is empty.*
   Large = size at least 3.
   Small = size at most 2.

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- *G has no stable set of size 3.*
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- *G has a vertex v with degree $d(v) < 12$.*

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- *G has a vertex v with degree $d(v) < 12$.*
- *$\omega(G) \geq 13$, and the sets R, X_1, X_4 are empty, and the sets X_2, X_3, X_5 are large.*

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Proof: Assume the first four items do not hold. Then:

If any X_i is small but not empty, then it contains a vertex of small degree.

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Proof: Assume the first four items do not hold. Then:

If any X_i is small but not empty, then it contains a vertex of small degree.

Hence each X_i is either large or empty.

Lemma

Suppose that $\omega(G) \geq 6$, and the sets R, X_1, X_4 are empty, and X_2, X_3, X_5 have size at least 2.
Then $\chi(G) = \omega(G)$ and an optimal coloring of G can be found in polynomial time.

Proof: By induction on $\omega(G)$.

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Proof: By induction on $\omega(G)$.

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- If $\omega(G) \geq 7$, we can find a stable set S that intersects all cliques of size $\omega(G)$.

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- If $\omega(G) = 6$, we can construct a 6-coloring directly.
- If $\omega(G) \geq 7$, we can find a stable set S that intersects all cliques of size $\omega(G)$. Then apply the algorithm to $G \setminus S$.

Conclusion and questions

Open cases for two excluded graphs of size 4:

1. (claw, $4P_1$)-free graphs.
2. (claw, $4P_1$, $2P_1 + P_2$)-free graphs.
3. (claw, $2P_1 + P_2$)-free graphs.
4. (C_4, $4P_1$)-free graphs.

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Another interesting open case:

- (P_k, triangle)-free for $k \leq 21$.

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6. (P_5, $K_{1,1,3}$)-free graphs,
7. (P_5, 4-wheel)-free graphs.

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6. (P_5, $K_{1,1,3}$)-free graphs,
7. (P_5, 4-wheel)-free graphs.

With T. Karthick and Lucas Pastor, we show that there is a polynomial-time algorithm for the first four classes.