Towards a Structure Theorem for Crossing-critical Graphs

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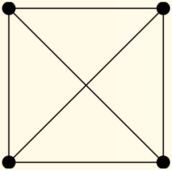
joint work with

Zdeněk Dvořák and **Bojan Mohar**

1 Drawing Graphs with Crossings

- The **crossing minimization problem**:



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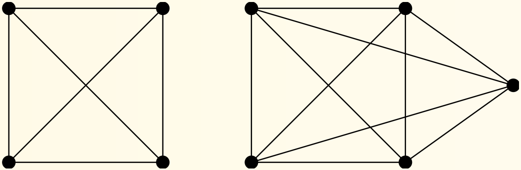

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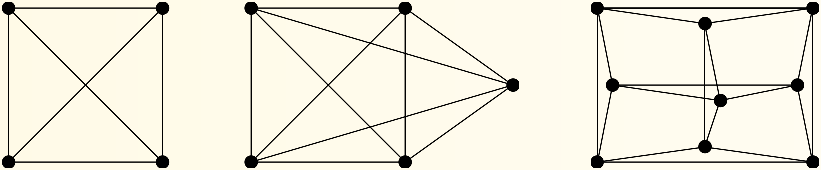

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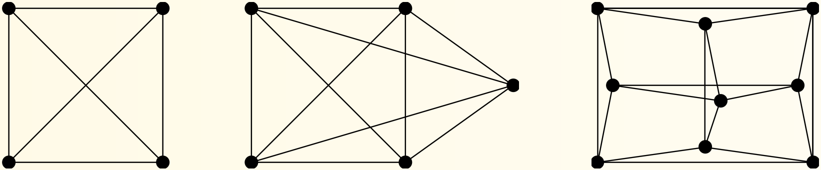

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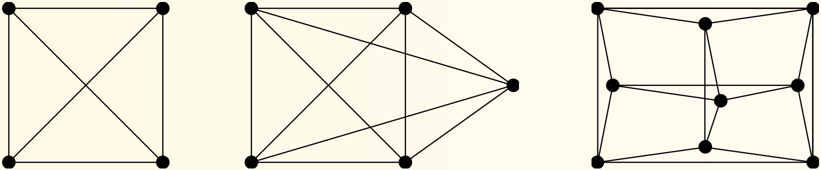

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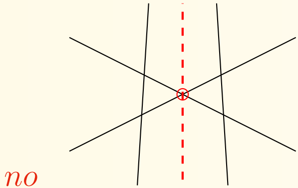

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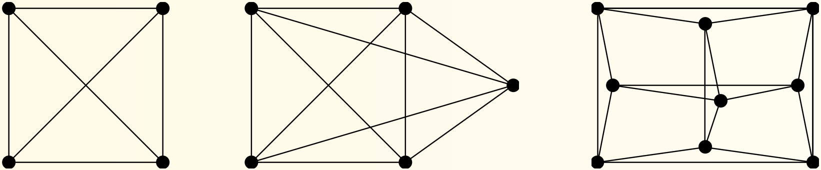

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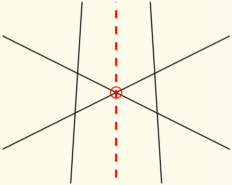
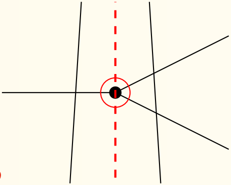
no  *no*  .

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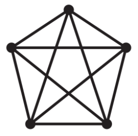
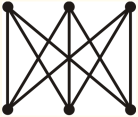
Definition. Graph H is *k-crossing-critical* if
$cr(H) \geq k$ and $cr(H-e) < k$ for all edges $e \in E(H)$.

We study crossing-critical graphs to understand what structural properties force the crossing number of a graph to be large.

Some "ancient" examples

- Kuratowski (30): The only 1-crossing-critical graphs K_5 and $K_{3,3}$.

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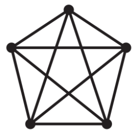

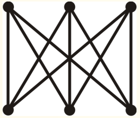

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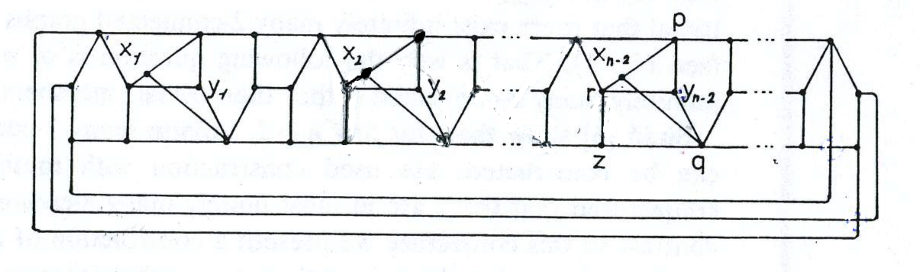
- Širáň (84), Kochol (87): Infinitely many k-crossing-critical graphs for every $k \geq 2$, even simple 3-connected.



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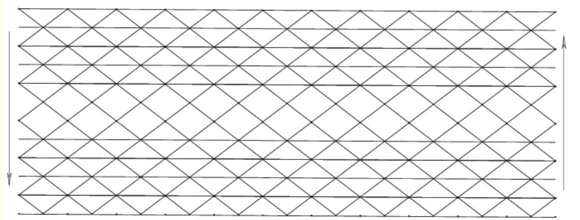

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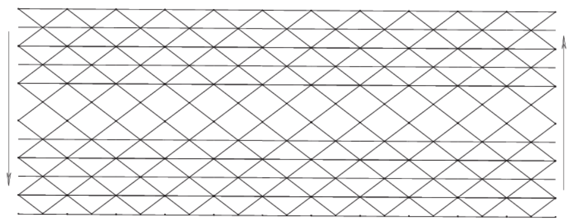
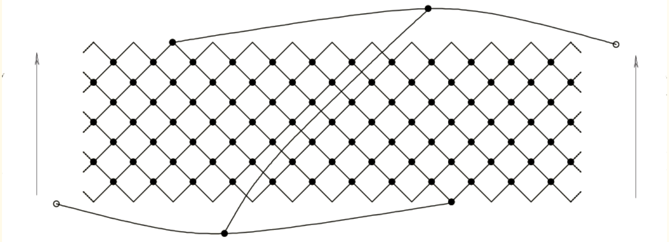


- PH (02):



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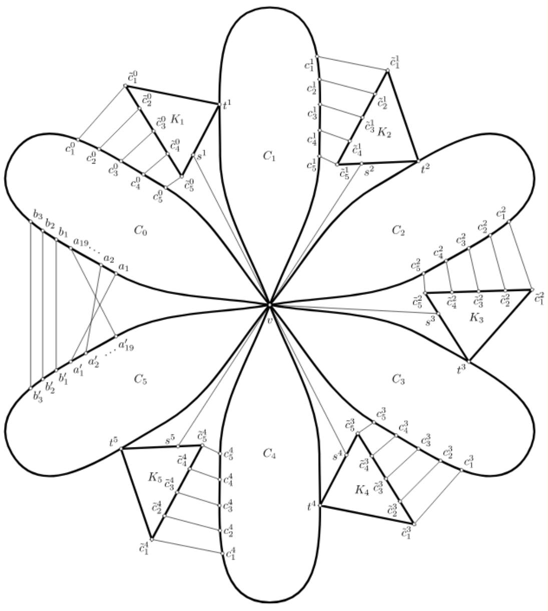
- Dvořák, PH, Mohar, Postle (11+):
  A k-crossing-critical graph cannot contain a deep nest, and so it has bounded dual diameter.

A bit of surprise

Dvořák, Mohar (10): A k-crossing-crit. graph with unb. degree, $k \geq 171$.



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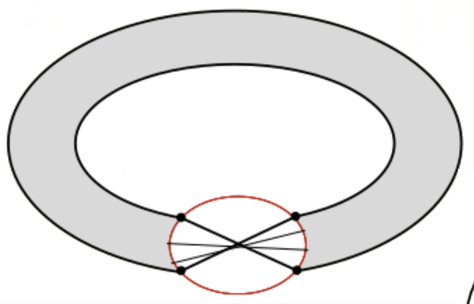

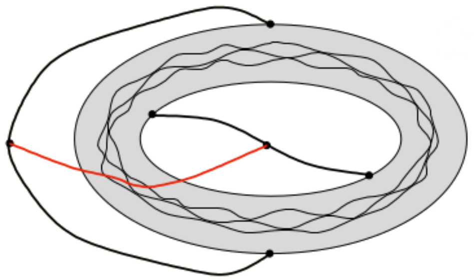

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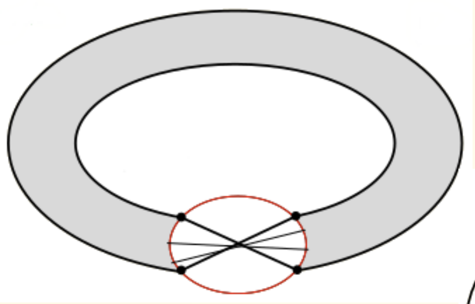
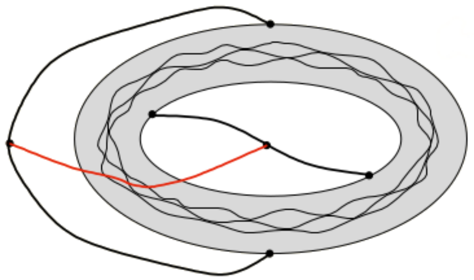
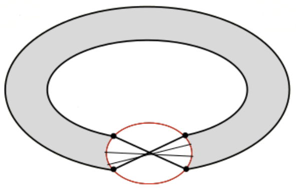
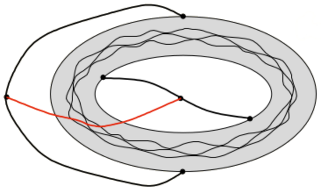




- Plus combinations of (fin. many) pieces like those in one graph.

4 Towards a Structural Description





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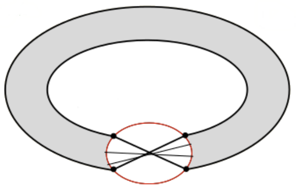

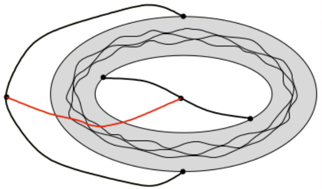

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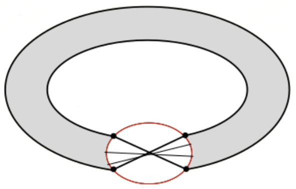

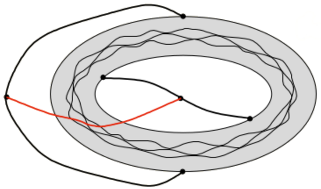

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3. There are finitely many well-defined building bricks that produce all k-crossing-critical graphs from a finite set of base graphs.

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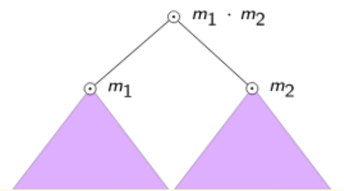

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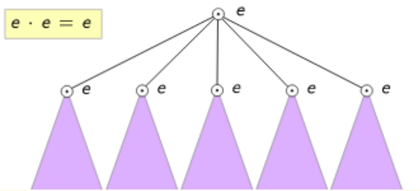

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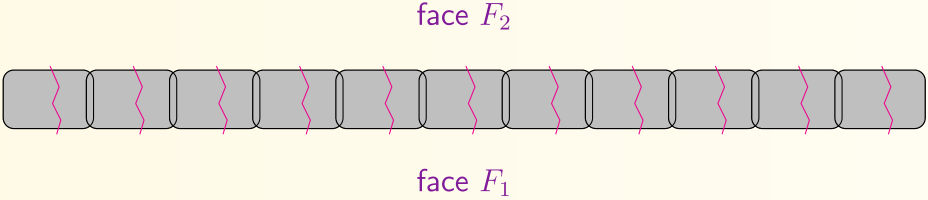

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