Exotic components of representation varieties for surface groups, and their Higgs bundle avatars

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Wokshop on Analysis of Gauge-Theoretic Moduli Spaces
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Disclaimer

These slides are precisely as they were during the talk on September 1, 2017. As such, they contain several omissions and inaccuracies, in both the mathematics and the attributions. Some of these, it must be admitted, are blemishes which reflect the author's limitations, but others reflect the fact that:

- The slides formed but one part of the talk. They were accompanied by verbal commentary designed to explain and embellish the contents of the slides
- This is a talk, not a paper. Any talk has to strike a balance between accuracy and accessibility. This balance inevitably involves the inclusion of some half-truths and/or white lies.

The author apologizes to anyone led astray by the inaccuracies or slighted in any way by the omissions.

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- The Topic
  - Moduli spaces associated to (S, G),
  - π_0: mundane versus ‘exotic’ contributions...
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[Joint with Brian Collier, Oscar Garcia-Prada, Peter Gothen, Andre Oliveira]

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- The Plan
  1. The moduli spaces
  2. Previously known sources of exotic components
  3. The new results

Setting the scene

(S, G)

(closed, oriented, $g > 1$)

($\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$, real forms)

$\mathrm{Rep}(\pi_1(S), G) \overset{NAH^*}{\longleftrightarrow} \mathcal{M}_{Higgs}(G)$

$\mathrm{Rep}(\pi_1(S), G) = \left(Hom^+(\pi_1(S), G)/G \right)$

$\mathcal{M}_{Higgs}(G) =$ (polystable G-Higgs bundles on $\Sigma = (S, J)$)

* Built from local systems, flat connections, harmonic metrics, and Hitchin-Kobayashi correspondences. [Hitchin, Simpson, Corlette, Donaldson,..]

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Focus for today: $\pi_0(\mathcal{M}_{Higgs}(G))$.

G-Higgs bundles on a Riemann surface $\Sigma = (S, J)$

For $G = \mathrm{SL}(n, \mathbb{C})$ a Higgs bundle on Σ is a pair (V, Φ)

- V = rank n holomorphic vector bundle with $\det(V) = \mathcal{O}$
- $\Phi : V \to V \otimes K_\Sigma$, holomorphic , $Tr(\Phi) = 0$

- Combined with a metric, h, on V the defining data define connections

$$\nabla_h = D_h + \Phi + \Phi^{*_h}$$

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[Hitchin, Simpson,....]

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[Hitchin, Simpson,....]

For $G \subset \mathrm{SL}(n,\mathbb{C})$: impose restrictions so ∇_h has holonomy in G

$G \subset \mathrm{SL}(n, \mathbb{C})$: Examples

G	V	Φ
$\mathrm{SL}(n, \mathbb{C})$	$\det(V_n) = \mathcal{O}$	$Tr(\Phi) = 0$
$\mathrm{SO}(n, \mathbb{C})$	$Q : V_n \times V_n \to \mathcal{O}$ (sym)	$\Phi^t Q + Q\Phi = 0$

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$\mathrm{SO}(n, \mathbb{C})$	$Q : V_n \times V_n \to \mathcal{O}$ (sym)	$\Phi^t Q + Q\Phi = 0$
$\mathrm{SL}(n, \mathbb{R})$	(V_n, Q)	$\Phi^t Q - Q\Phi = 0$
$\mathrm{SO}(p, q)$	$(V_{p+q}, Q) = (V_p, Q_p) \oplus (W_q, Q_q)$	$\Phi = \begin{bmatrix} 0 & \eta \\ -\eta^T & 0 \end{bmatrix}$

$\mathcal{M}(G)$ for complex G $(\mathrm{SL}(n,\mathbb{C}), \mathrm{SO}(n,\mathbb{C}), \mathrm{Sp}(2n,\mathbb{C})..)$

Interesting

1. hyperkahler structure
2. [$\mathbb{C}^*$-action] $\lambda[V,\Phi] = [V,\lambda\Phi]$
3. ['Morse function'] $f[V,\Phi] = ||\Phi||^2_{L^2}$

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4. [Hitchin fibration]

$$H : \mathcal{M}_{Higgs}(G) \xrightarrow{\{p_1 \ldots p_r\}} \mathbb{C}^N = \bigoplus_{i=1}^r H^0(K^{d_i})$$

$[V,\Phi] \mapsto (p_1(\Phi) \ldots p_r(\Phi))$

- generic fibers are abelian varieties
- setting for mirror symmetry

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Less interesting

$\pi_0(\mathcal{M}_{Higgs}(G))$ is determined by topological types of the bundles

[Li, Garcia-Prada–Oliveira]

$\mathcal{M}(G)$ for real forms $(\mathrm{SL}(n,\mathbb{R}), \mathrm{SO}(p,q)...)$

Same as for G complex:

- $\mathcal{M}_{Higgs}(G)$ is a moduli space of bundles (with extra structure) on S
- Bundles have discrete invariants, (c) labelling topological type

$$\mathcal{M}_{Higgs}(G) = \coprod_c \mathcal{M}_c$$

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New phenomena:

- (Too many invariants) Bounds can constrain the values of the invariants
- (Too few invariants) Topological invariants don't tell the whole story
- (Special components) Seen in corresponding components of $\mathrm{Rep}(\pi_1(\Sigma), G)$

$\mathrm{Rep}(\pi_1(S), \mathrm{SL}(2,\mathbb{R}))$ [Goldman '88]

- $\mathrm{Rep}(\pi_1(S), \mathrm{SL}(2,\mathbb{R}))$ has $2^{2g+1} + 2g - 3$ components
- The components are labelled by integer $|e| \leq g - 1$ [Milnor-Wood]
- For $|e| < g - 1$ the components are connected
- At $e = \pm(g-1)$: 2^{2g} copies of $\mathcal{T}$ (Teichmuller space)

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....what about $\mathcal{M}(\mathrm{SL}(2,\mathbb{R}))$? [Hitchin, '87]

$\mathcal{M}(\mathrm{SL}(2,\mathbb{R}))$

$$V = L \oplus L^{-1} \quad \Phi = \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix} \qquad \begin{matrix} \beta \in H^0(L^2 K) \\ \gamma \in H^0(L^{-2} K) \end{matrix}$$

- Topological invariant: $\tau = \deg(L)$
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$$\boxed{\mathcal{M}_{\pm(g-1)} = \coprod_{K^{1/2}} \mathcal{M}^{Hitch}_{K^{1/2}}} \qquad \mathcal{M}^{Hitch}_{K^{1/2}} \simeq H^0(K^2)$$

(2^{2g} Hitchin/Teichmuller components)

General phenomena

$$Sp(2,\mathbb{R}) = SL(2,\mathbb{R}) \xrightarrow{2:1} PSL(2,\mathbb{R}) = SO_0(1,2)$$

$$Sp(2n,\mathbb{R}) \qquad SL(n,\mathbb{R}) \qquad SO_0(2,q), SO_0(p,p+1)$$

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- Split real forms
- Hermitian, i.e. G/K is non-compact Hermitian symmetric
- Both

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- Split real forms
- G/K is Hermitian symmetric (non-compact)
- Both
- ???

$$\mathrm{Rep}(\pi_1(S), G_{split}) \supset \mathcal{T}(G_{split}) = \mathcal{M}^{Hitch}(G_{split}) \subset \mathcal{M}(G_{split})$$

Split real forms $(\mathrm{SL}(n,\mathbb{R}), \mathrm{SO}_0(p,p), \mathrm{SO}_0(p,p+1), \mathrm{Sp}(2n,\mathbb{R}))$ $\left(K^{1/2} \oplus K^{-1/2}, \begin{bmatrix} 0 & \beta \\ 1 & 0 \end{bmatrix}\right)$

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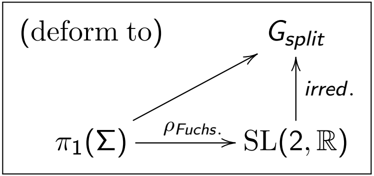

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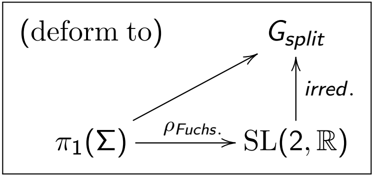

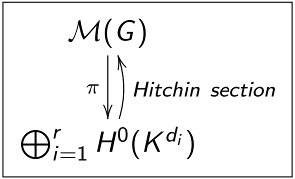

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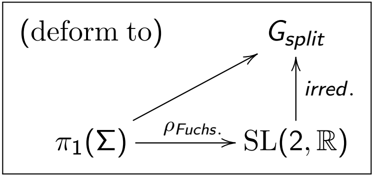
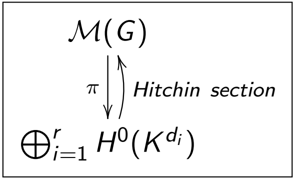
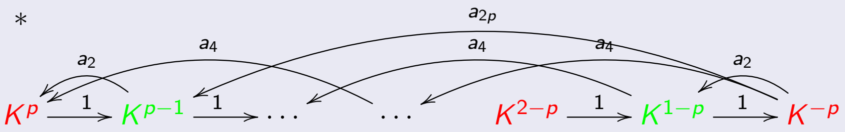




$\mathcal{M}^{Hitch}(\mathrm{SO}(p,p+1)) \simeq \bigoplus_{j=1}^{p} H^0(K^{2j})$

$$\left(V, W, \begin{bmatrix} 0 & \eta \\ -\eta^T & 0 \end{bmatrix}\right) = \left(\bigoplus_{j=0}^{p-1} K^{p-1-2j}, \bigoplus_{j=0}^{p} K^{p-2j}, *\right)$$

$*$



$$\mathcal{M}(G) = \coprod_{|\tau| \leq \tau_{max}} \mathcal{M}_\tau$$

$$\boxed{\mathcal{M}_{\tau_{max}} = \coprod_{c'} \mathcal{M}_{max,c'}}$$

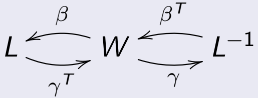
$$\mathcal{M}_{max,c'} = \mathcal{M}^{K^2}_{c'}(G')$$

- $\tau \in \kappa\mathbb{Z}$ is topological; τ_{max} is a Milnor-Wood bound
- c' is a new topological invariant
- G' is called the Cayley partner group

[B., Collier, Garcia-Prada, Gothen, Mundet, Oliveira, Biquard–Garcia-Prada–Rubio]

$\mathrm{SO}_0(2,q)$-Higgs bundles $\left((V_2,Q_2),(W_q,Q_q),\begin{bmatrix}0 & \eta\\ -\eta^T & 0\end{bmatrix}\right)$

- $(V_2,Q_2)=\left(L\oplus L^{-1},\ \begin{bmatrix}0&1\\1&0\end{bmatrix}\right),\ \eta=\begin{bmatrix}\beta\\ \gamma\end{bmatrix}$ with $\begin{cases}\beta : W\to LK\\ \gamma : W\to L^{-1}K\end{cases}$



- Invariants: $(\tau=\deg L, w=w_2(W))\in\mathbb{Z}\times\mathbb{Z}_2$
- Stability $\implies$ $L\xrightarrow{\gamma\gamma^T}L^{-1}K^2$ not zero
- Bound: $|\tau|\leq 2g-2$

$\mathrm{SO}_0(2,q)$-Higgs bundles

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$$L \underset{\gamma^T}{\overset{\beta}{\rightleftarrows}} W \underset{\gamma}{\overset{\beta^T}{\rightleftarrows}} L^{-1}$$

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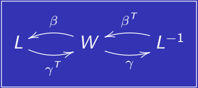
$$\mathcal{M}(\mathrm{SO}(2,q))=\coprod_{|\tau|\le 2g-2,w}\mathcal{M}_{\tau,w}$$

Maximal $\mathrm{SO}_0(2, q)$-Higgs bundles ($\tau = 2g - 2$:)

$$L \overset{\beta}{\underset{\gamma^T}{\leftrightarrows}} W \overset{\beta^T}{\underset{\gamma}{\leftrightarrows}} L^{-1}$$

- $\gamma^T\gamma : L \to L^{-1}K$ an isom.



- $\gamma^T\gamma : L \to L^{-1}K$ an isom.
- $L = IK$ where $I^2 = \mathcal{O}$
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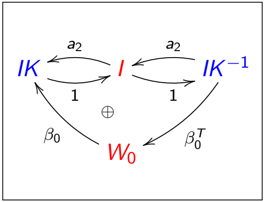

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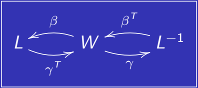

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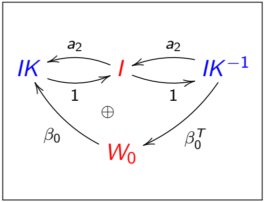

- $\boxed{\beta_0 : W_0 \to IK^2}$ defines $[I, W_0, \beta_0] \in \mathcal{M}^{K^2}_{I,w}(SO(1,q-1))$
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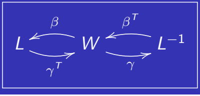

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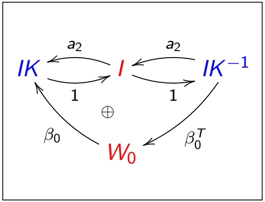

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$$\mathcal{M}_{\tau_{max},w}(\mathrm{SO}(2,q)) = \coprod_{I^2=\mathcal{O}} \mathcal{M}^{K^2}_{I,w}(SO(1,q-1)) \times H^0(K^2)$$

$$= \coprod\nolimits_{I^2=\mathcal{O}} \mathcal{M}^{K^2}_{I,w}(SO(1,q-1) \times \mathrm{SO}(1,1)) = \coprod\nolimits_{I^2=\mathcal{O}} \mathcal{M}^{K^2}_{I,w}(G')$$

Theorem (BCGGO)

Fix $2 \leq p \leq q$. The moduli space $\mathcal{M}(\mathrm{SO}(p,q))$ has exotic components whose union is isomorphic to

$$\mathcal{M}^{exotic}(\mathrm{SO}(p,q)) = \mathcal{M}^{K^p}(\mathrm{SO}(1, q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j})$$

- $2 < p < q-1$: neither split nor Hermitian
- $\mathcal{M}^{exotic}(\mathrm{SO}(p,q))$ not detected by obvious topological invariants

$$\boxed{(V, W, \eta)}$$

$$\boxed{\begin{array}{l} a = sw_1(V) = sw_1(W); \\ b = sw_2(V); \ c = sw_2(W) \end{array}}$$

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$\mathrm{SO}(p,q)$: more precisely

$$(V, W, \eta)$$

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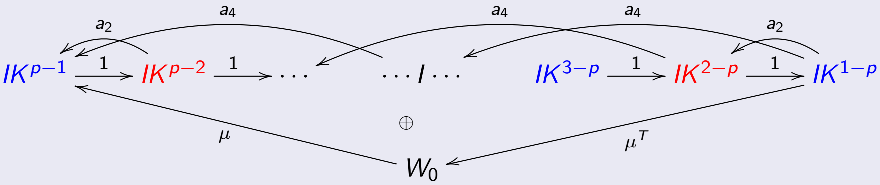

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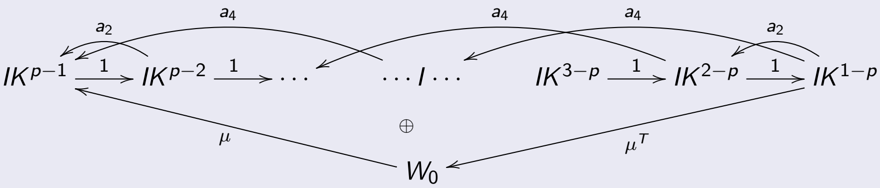

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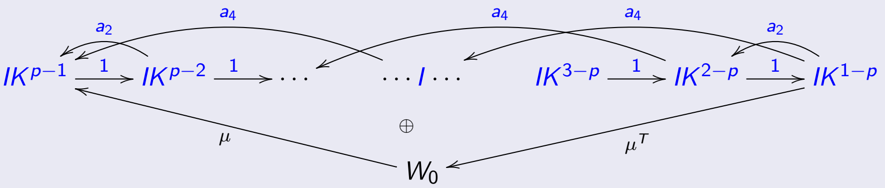

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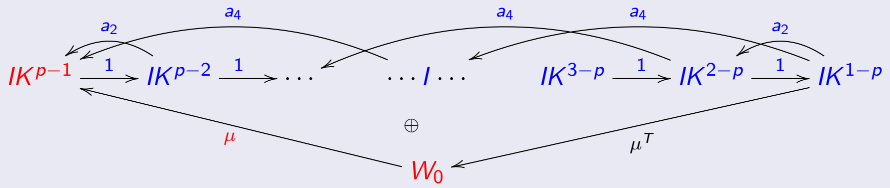

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Evidence from Morse theory

[Hitchin] $f : \mathcal{M}(G) \to \mathbb{R}$ defined by $f([E, \Phi]) = ||\Phi||^2$

- f proper $\implies$ attains local min on all connected components

$f : \mathcal{M}_{a,b,c}(\mathrm{SO}(p,q)) \to \mathbb{R}$ defined by $f([V, W, \eta]) = ||\eta||^2$

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Theorem (Arroyo 2009)

For each (a, c) f has local minima on $\mathcal{M}_{a,0,c}$ with $\eta \neq 0$ and of the form

$$IK^{p-1} \xrightarrow{1} IK^{p-2} \xrightarrow{1} \cdots \qquad \cdots I \cdots \qquad IK^{3-p} \xrightarrow{1} IK^{2-p} \xrightarrow{1} IK^{1-p}$$

$$\oplus$$

$$W_0$$

(where $a = sw_1(W_0) \sim I$ and $c = w_2(W_0)$)

The case $\mathrm{SO}(p, p+1)$ $(q - p + 1 = 2)$

Theorem (Collier '16)

For each integer $d \in [0, 2p(g-1)]$ *there is an exotic connected component of* $\mathcal{M}(\mathrm{SO}(p, p+1))$

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- At $0 < d < p(2g-2)$ these are diffeomorphic to $\mathcal{F}_d \times \bigoplus_{j=1}^{p-1} H^0(K^{2j})$

where

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- $$\coprod_{d=c(mod 2)} \mathcal{F}_d = \mathcal{M}^{K^p}_{0,c}(\mathrm{SO}(1,2))$$

Guichard-Wienhard Positivity

- Hitchin representations (G split) and maximal representations (G Hermitian) share Labourie's Anosov property
- The Anosov property is related to a positivity property enjoyed by the groups

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NAH: $\mathrm{Rep}(\pi_1(S), G) = \mathcal{M}(G)$

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- The groups $\mathrm{SO}(p, q)$ are the only other (non-exceptional) groups which allow positive representations

Insights from spectral data

$$h : \mathcal{M}(\mathrm{SO}(p,q)) \to \bigoplus_{j=1}^{p} H^0(K^{2j})$$

- An alternative description of generic fibers of Hitchin fibration - see the previous talk by Laura Schaposnik!
- Though not sufficient to distinguish connected components, spectral data sheds new light on
  - their intersection with regular fibers, and hence
  - the invariants which label the components, and
  - the structure of the components

[David Baraglia and Laura Schaposnik]

What we actually prove

We want:

$$\Psi : \mathcal{M}^{K^p}(\mathrm{SO}(1, q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}) \to \mathcal{M}(\mathrm{SO}(p,q))$$

$$\left(\left[I, W_0, \begin{bmatrix} 0 & \mu \\ -\mu^T & 0 \end{bmatrix}\right], (a_2, a_4, \ldots, a_{2p-2})\right) \longmapsto \left[V, W, \begin{bmatrix} 0 & \eta \\ -\eta^T & 0 \end{bmatrix}\right]$$

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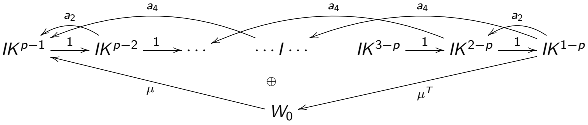

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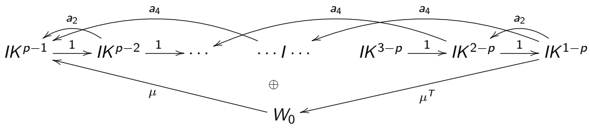

We show:

- $\tilde{\Psi}$ descends to moduli spaces
- Ψ is closed
- Ψ is open

What we actually prove

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1. $\tilde{\Psi}$ descends: check stability conditions and equivariance

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2. Ψ is closed: use Hitchin fibrations to understand divergent sequences

What we actually prove

$$\tilde{\Psi}\left(\left(I, W_0, \begin{bmatrix} 0 & \mu \\ -\mu^T & 0 \end{bmatrix}\right), \vec{a}\right) = \left(V, W, \begin{bmatrix} 0 & \eta \\ -\eta^T & 0 \end{bmatrix}\right)$$

1. $\tilde{\Psi}$ descends: check stability conditions and equivariance
2. Ψ is closed: use Hitchin fibrations to understand divergent sequences
3. Ψ is open: use Kuranishi slices, Hitchin sections, mildness of singularities to show that the linearization is injective.

Conjecture

All representations in components of $\mathrm{Rep}(\pi_1(S), \mathrm{SO}(p, q))$ corresponding to exotic components of $\mathcal{M}(\mathrm{SO}(p, q))$ satisfy the Guichard-Wienhard positivity condition.

- True at representations corresponding to local minima of the Hitchin function ($||\Phi||^2$).