Gradient Flows, Heat Equation, and Brownian Motion on Time-dependent Metric Measure Spaces

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Outline

- Heat flow as gradient flow for the entropy
- Dirichlet heat flow
- Heat flow on time-dependent metric measure spaces
- (Super-) Ricci flows

Heat Flow on Metric Measure Spaces

(X, d) complete separable metric space, m locally finite measure

Heat equation on X

- either as gradient flow on $L^2(X, m)$ for the **energy**

$$\mathcal{E}(u) = \frac{1}{2}\int_X |\nabla u|^2 \, dm = \liminf_{\substack{v \to u \text{ in } L^2(X,m) \\ v \in Lip(X,d)}} \frac{1}{2}\int_X (\mathrm{lip}_x v)^2 \, dm(x)$$

  with $|\nabla u|$ = minimal weak upper gradient

- or as gradient flow on $\mathcal{P}_2(X, d)$ for the **relative entropy**

$$\mathrm{Ent}(u\, m) = \int_X u \log u \, dm.$$

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Theorem (Ambrosio/Gigli/Savare).

For arbitrary metric measure spaces (X, d, m) satisfying $CD(K, \infty)$ both approaches coincide.

Note: Heat flow $P_t u$ not necessarily linear.

Synthetic Ricci Bounds for Metric Measure Spaces

(X, d) complete separable metric space, m locally finite measure

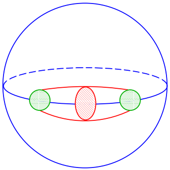
Definition. The Curvature-Dimension Condition $CD(K, \infty)$

$\iff \quad \forall \mu_0, \mu_1 \in \mathcal{P}(X): \; \exists$ geodesic $(\mu_t)_t$ s.t. $\forall t \in [0,1]$:

$$S(\mu_t) \leq (1-t)\, S(\mu_0) + t\, S(\mu_1) - \frac{K}{2}\, t(1-t)\, W^2(\mu_0, \mu_1)$$

with Boltzmann entropy

$$S(\mu) = \mathrm{Ent}(\mu|m) = \begin{cases} \int_X \rho \log \rho \, dm & , \text{ if } \mu = \rho \cdot m \\ +\infty & , \text{ if } \mu \not\ll m \end{cases}$$



and Kantorovich-Wasserstein metric

$$W(\mu_0, \mu_1) = \inf_q \left\{ \int_{X \times X} d^2(x,y)\, d\,q(x,y) : \quad (\pi_1)_* q = \mu_0, \; (\pi_2)_* q = \mu_1 \right\}^{1/2}$$

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or equivalently

$$\partial_t S(\mu_t)\Big|_{t=1} - \partial_t S(\mu_t)\Big|_{t=0} \geq K \cdot W^2(\mu_0, \mu_1)$$

with Boltzmann entropy

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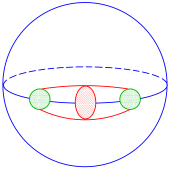

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The Relevance of Lower Ricci Bounds

Nonnegative Ricci curvature implies that – in many respects – optimal transports, heat flows, and Brownian motions behave as nicely as on Euclidean spaces. For instance

- Heat kernel comparison

$$p_t(x,y) \geq (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x,y)}{4t}\right)$$

- Li-Yau estimates
- Gradient estimates

$$|\nabla P_t u| \leq P_t(|\nabla u|)$$

- Transport estimates

$$W(P_t\mu, P_t\nu) \leq W(\mu,\nu)$$

- $\forall x,y:\ \exists$ coupled Brownian motions $(X_t, Y_t)_{t\geq 0}$ starting at (x,y) s.t. $\mathbb{P}$-a.s. for all $t \geq 0$

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Indeed, $\mathrm{Ric} \geq 0$ is necessary and sufficient for each of the latter properties.

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Among the applications:
'Market Fragility, Systemic Risk, and Ricci Curvature' (Sandhu et al. 2015)
'Ricci curvature and robustness of cancer networks' (Tannenbaum et al. 2015)

The Gradient Flow Perspective

Powerful consequences of the gradient flow perspective (Otto, Villani)

$$\mathrm{Hess}\, S \geq K$$

$$\Downarrow$$

$$|\nabla S|^2 \geq 2\, K \cdot S$$

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$$S \geq K/2 \cdot W_2(\cdot, \nu_\infty)^2$$

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$$\mathrm{Hess}\, S \geq K \qquad \text{"Bakry-Emery inequality"}$$

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$$|\nabla S|^2 \geq 2\, K \cdot S \qquad \text{"log. Sobolev inequality"}$$

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$$S \geq K/2 \cdot W_2(\cdot, \nu_\infty)^2 \qquad \text{"Talagrand inequality"}$$

Dirichlet Heat Flow

(X, d) complete separable metric space, $Y \subset X$ open with $\emptyset \neq Y \neq X$.

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Charged probability measures on X

$$\sigma = (\sigma^+, \sigma^-) : \ \sigma^\pm \in \mathcal{P}^{sub}(X), \ \sigma^+|_{X \setminus Y} = \sigma^-|_{X \setminus Y}, \sigma^+(X) + \sigma^-(X) = 1;$$

Effective measure $\sigma^0 = \sigma^+ - \sigma^-$, *total measure* $\overline{\sigma} = \sigma^+ + \sigma^-$.

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Kantorovich-Wasserstein distance between two charged probability measures $\sigma = (\sigma^+, \sigma^-)$ and $\tau = (\tau^+, \tau^-)$

$$\begin{aligned} \hat{W}_2(\sigma, \tau)^2 \quad = \quad & \inf \Big\{ \int \int d(x,y)^2 dq^{++}(x,y) + \int \int d^*(x,y)^2 dq^{+-}(x,y) \\ & + \int \int d^*(x,y)^2 dq^{-+}(x,y) + \int \int d(x,y)^2 dq^{--}(x,y) : \\ & \sigma^i = \sigma^{i+} + \sigma^{i-}, \tau^j = \tau^{+j} + \tau^{-j}, q^{ij} \in \mathrm{Cpl}(\sigma^{ij}, \tau^{ij}), i,j \in \{+,-\} \Big\} \end{aligned}$$

where

$$d^*(x,y) := \inf_{z \in X\setminus Y} \big[d(x,z) + d(z,y) \big].$$

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Thm. (Profeta, St. '17) Assume X Riem. mfd. with $\mathrm{Ric} \geq K$, Y convex subset. Then Dirichlet heat semigroup $(P_t^0)_{t>0}$ on Y is given by the effective measure of the gradient flow for the Boltzmann entropy $\mathrm{Ent}(\sigma^+|m)+\mathrm{Ent}(\sigma^-|m)$ within the space of charged prob. measures w.r.t. the distance $\hat{W}_2$.

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Cor.1 $\forall$ subprobab. μ, ν on Y

$$\overset{o}{W}_2\,(P_t^0\mu, P_t^0\nu) \leq e^{-Kt} \cdot \overset{o}{W}_2\,(\mu, \nu)$$

with Kantorovich-Wasserstein distance between subprobabilities $\mu, \nu \in \mathcal{P}^{sub}(Y)$

$$\begin{aligned}
\overset{o}{W}_2\,(\mu, \nu) &= \inf\left\{\hat{W}_2\big((\sigma^+, \sigma^-), (\tau^+, \tau^-)\big) : \ \sigma^+ - \sigma^- = \mu, \tau^+ - \tau^- = \nu\right\} \\
&= \inf\Big\{\hat{W}_2\big((\mu + \rho, \rho), (\nu + \eta, \eta)\big) : \ \rho, \eta \in \mathcal{P}^{sub}(X), \\
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Cor.2 $\big|\nabla P_t^0 u\big| \leq e^{-Kt}\cdot P_t^N |\nabla u|$

Heat Flow on Time-dependent MM-Spaces

Aim. Study heat flow on $I \times X$ where $I = (0, T) \subset \mathbb{R}$ and (X, d_t, m_t) is metric measure space $(\forall t \in I)$

Many challenges.

- Define/study solutions for heat equation $\partial_t u = \Delta_t u$
- Define/study gradient flows for energy and for entropy
- Find correct time-dependent versions of Bakry-Emery (=Bochner) and of Lott-St.-Villani conditions
- Establish equivalence between Eulerian and Lagrangian approach

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$$\frac{1}{2}\partial_t |x_t - z|^2 \leq V(z) - V(x_t) \qquad (\forall z)$$

Evolution variational inequality (in Hilbert spaces) for ‘static’ V

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Powerful extension to 'static' metric spaces (Ambrosio, Gigli, Savaré)

Gradient Flows on Time-dependent MM-Spaces

Question: How to define gradient flow for time-dependent potential $V : I \times X \to (-\infty, \infty]$ on time-dependent metric space $(X, d_t)_{t \in I}$?

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Examples:

- gradient flow for Boltzmann entropy S_t on $(\mathcal{P}, W_t)_{t \in I}$
- gradient flow for Dirichlet energy $\mathcal{E}_t$ on $L^2(X, m_t)_{t \in I}$

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Definition

An absolutely continuous curve $(x_t)_{t \in J}$ will be called *dynamic backward EVI^--gradient flow* for V if for all $t \in J$ and all $z \in Dom(V_t)$

$$\frac{1}{2} \partial_s^- d_{s,t}^2(x_s, z)\Big|_{s=t-} \geq V_t(x_t) - V_t(z)$$

where

$$d_{s,t}(x,y) := \inf \left\{ \int_0^1 |\dot{\gamma}^a|^2_{s+a(t-s)} da \right\}^{1/2}$$

with infimum over all absolutely continuous curves $(\gamma^a)_{a \in [0,1]}$ in X from x to y.

Heat Flow on Time-dependent MMS

For the sequel, a 1-parameter family of metric measure spaces (X, d_t, m_t), $t \in I \subset \mathbb{R}$ will be given s.t. $\forall s, t \in I$

- the mm-space (X, d_t, m_t) satisfies $\mathrm{CD}^*(K, N)$ and has linear heat flow
- $\log \frac{d_t(x,y)}{d_s(x,y)} \leq C\,|s-t|$
- $m_t(dx) = e^{-f_t(x)} m_0(dx)$ for some $f \in Lip(I \times X)$

Thus $\forall t \in I$:
$\exists$ Dirichlet form $\mathcal{E}_t$, Laplacian Δ_t, squared gradient $\Gamma_t(u) = |\nabla_t u|^2$.

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Theorem ('Heat equation')

$\exists$ heat kernel p on $\{(t, s, x, y) \in I^2 \times X^2 : t > s\}$, Hölder continuous in all variables and satisfying the propagator property $p_{t,r}(x, z) = \int p_{t,s}(x, y) p_{s,r}(y, z)\, dm_s(y)$, such that

- $(t, x) \mapsto p_{t,s}(x, y)$ solves the heat equation $\partial_t u_t = \Delta_t u_t$ on $(s, T) \times X$
- $(s, y) \mapsto p_{t,s}(x, y)$ solves the adjoint heat equation $\partial_s v_s = -\Delta_s v_s + (\partial_s f_s) \cdot v_s$ on $(0, t) \times X$

Heat Flow on Time-dependent MMS

(i) $\forall s \in I, \forall h \in L^2(X, m_s)$: $\exists!$ solution to heat equation $\partial_t u_t = \Delta_t u_t$ on $(s, T) \times X$ with $u_s = h$ given by

$$u_t(x) = P_{t,s}h(x) := \int p_{t,s}(x,y)h(y)\,dm_s(y)$$

(ii) $\forall t \in I, \forall g \in L^2(X, m_t)$: $\exists!$ solution to the adjoint heat equation $\partial_s v_s = -\Delta_s v_s + (\partial_s f_s) \cdot v_s$ on $(0, t) \times X$ with $v_t = g$ given by

$$v_s(y) = P^*_{t,s}g(y) := \int p_{t,s}(x,y)g(x)\,dm_t(x)$$

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(iii) Define dual heat flow $\hat{P}_{t,s} : \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$(\hat{P}_{t,s}\mu)(dy) = \left[\int p_{t,s}(x,y)\, d\mu(x) \right] m_s(dy).$$

In particular, $\hat{P}_{t,s}\big(g \cdot m_t\big) = \big(P^*_{t,s} g\big) \cdot m_s$ and

$$\int h d\left(\hat{P}_{t,s}\mu\right) = \int \left(P_{t,s} h\right) d\mu$$

Heat Flow and its Dual as Gradient Flows

Theorem. Assume $m_t \leq e^{L(t-s)} m_s$.

$\forall u \in Dom(\mathcal{E}), \forall s$ the heat flow $t \mapsto u_t = P_{t,s}u$ is the unique dynamical forward EVI_L-gradient flow for the energy $\frac{1}{2}\mathcal{E}$, that is, $\forall v \in Dom(\mathcal{E}), \forall t$

$$-\frac{1}{2}\partial_s^- \|u_s - v\|_{s,t}^2\Big|_{s=t-} + \frac{L}{4}\|u_t - v\|_t^2 \geq \frac{1}{2}\mathcal{E}_t(u_t) - \frac{1}{2}\mathcal{E}_t(v).$$

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Theorem. Assume that $(X, d_t, m_t)_{t\in I}$ is a super-Ricci flow.

$\forall \mu \in Dom(S), \forall T$ the dual heat flow $s \mapsto \mu_s = \hat{P}_{T,s}\mu$ is the unique dynamical backward EVI-gradient flow for the Boltzmann entropy S, that is, $\forall \sigma \in Dom(S), \forall t$

$$\frac{1}{2}\partial_s^- W_{s,t}^2(\mu_s, \sigma)\Big|_{s=t-} \ge S_t(\mu_t) - S_t(\sigma).$$

Here $W_{s,t}^2(\mu_0,\mu_1) := \inf \int_0^1 |\dot{\mu}^a|_{s+a(t-s)}^2 da$ with infimum over all $\mathcal{AC}^2$-curves $(\mu^a)_{a\in[0,1]}$ in $\mathcal{P}(X)$ connecting μ^0 and μ^1.

Super Ricci Flows

A family of Riemannian manifolds (M, g_t), $t \in (0, T)$, is called super-Ricci flow iff

$$\mathrm{Ric}_t + \frac{1}{2}\partial_t g_t \geq 0.$$

Two main examples

- Static manifolds with $\mathrm{Ric} \geq 0$ (‘elliptic case’)
- Ricci flows $\mathrm{Ric}_t = -\frac{1}{2}\partial_t g_t$ (‘minimal super-Ricci flows’)

Hamilton’81, Perelman’02, . . . , McCann/Topping’10, Lott’09, Arnaudon/Coulibaly/Thalmaier’08, Kuwada/Philipowski’11, X.-D.Li’14, Kleiner/Lott’14, Haslhofer/Naber’15

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Given a 1-parameter family of metric measure spaces (X, d_t, m_t), $t \in I \subset \mathbb{R}$. Consider the function

$$S: \ I \times \mathcal{P}(X) \to (-\infty, \infty], \quad (t, \mu) \mapsto S_t(\mu) = \mathrm{Ent}(\mu | m_t)$$

where $\mathcal{P}(X)$ is equipped with the 1-parameter family of metrics W_t (= L^2-Wasserstein metrics w.r.t. d_t).

Definition.

$(X, d_t, m_t)_{t \in I}$ is super-Ricci flow iff for a.e. t and every W_t-geodesic $(\mu^a)_{a \in [0,1]}$

$$\partial_a S_t(\mu^a)\big|_{a=0} - \partial_a S_t(\mu^a)\big|_{a=1} \leq \frac{1}{2}\partial_t^- W_{t-}^2(\mu^0, \mu^1).$$

Characterization of Super-Ricci Flows

Theorem. The following are equivalent

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- $W_s(\hat{P}_{t,s}\mu, \hat{P}_{t,s}\nu) \leq W_t(\mu, \nu)$

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- $W_s(\hat{P}_{t,s}\mu, \hat{P}_{t,s}\nu) \leq W_t(\mu, \nu)$
- $\forall x, y, \forall t :\ \exists$ coupled backward Brownian motions $(X_s, Y_s)_{s \leq t}$ starting at t in (x, y) s.t. $d_s(X_s, Y_s) \leq d_t(x, y)$ a.s. for all $s \leq t$

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- $|\nabla_t(P_{t,s}u)|^2 \leq P_{t,s}(|\nabla_s u|^2)$

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- $W_s(\hat{P}_{t,s}\mu, \hat{P}_{t,s}\nu) \leq W_t(\mu, \nu)$
- $\forall x, y, \forall t: \ \exists$ coupled backward Brownian motions $(X_s, Y_s)_{s \leq t}$ starting at t in (x, y) s.t. $d_s(X_s, Y_s) \leq d_t(x, y)$ a.s. for all $s \leq t$
- $|\nabla_t(P_{t,s}u)|^2 \leq P_{t,s}(|\nabla_s u|^2)$
- $\Gamma_{2,t} \geq \frac{1}{2}\partial_t \Gamma_t$ where $\Gamma_{2,t}(u) = \frac{1}{2}\Delta_t|\nabla_t u|^2 - \langle \nabla_t u, \nabla_t \Delta_t u\rangle$

Functional Inequalities for Super-Ricci Flows

Theorem. For every super-Ricci flow $(X, d_t, m_t)_{t\in I}$

- Local Poincare inequalities

$$2(t-s)\Gamma_t(P_{t,s}u) \le P_{t,s}(u^2) - (P_{t,s}u)^2 \le 2(t-s)P_{t,s}(\Gamma_s u)$$

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- Dimension-free Harnack inequality: $\forall \alpha > 1$

$$(P_{t,s}u)^\alpha(y) \le P_{t,s}u^\alpha(x) \cdot \exp\left(\frac{\alpha d_t^2(x,y)}{4(\alpha-1)(t-s)}\right)$$

Characterization of Ricci Flows

Def.

$(X, d_t, m_t)_{t \in I}$ is a Ricci flow iff super-Ricci flow and $\forall x, t$

$$\lim_{y \to x} \partial_s \log W_s\left(\hat{P}_{t,s}\delta_x, \hat{P}_{t,s}\delta_y\right)\Big|_{s=t-} = 0$$

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Cor.

If $(X, d_t, m_t)_{t\in I}$ is Ricci flow then $\forall t, \forall d_t$-geodesics $(\gamma^a)_{a\in[0,1]}$ the length of the curve

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Theorem.

Assume that X is N-cone of some mm-space Y. Then X is Ricci bounded if and only if $N \in \mathbb{N}$ and $X = \mathbb{R}^{N+1}$.

Thank You For Your Attention!

Synthetic Upper Bounds for Ricci Curvature

Theorem

For weighted Riem with $\mathrm{Ric}_\infty > -K_0$, for all non-conjugate x, y

$$K_{x,y} \leq -\partial_t^- \log W\left(P_t\delta_x, P_t\delta_y\right)\Big|_{t=0+} \leq K_{x,y} + \sigma_{x,y} \tan^2\left(\sqrt{\sigma_{x,y}}d(x,y)/2\right)$$

with $K_{x,y}$ = average Ricci curvature along min. geodesic from x to y
and $\sigma_{x,y}$ = maximal modulus of sect. curv. along this geodesic.

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Singular examples

- Doubling of $\overline{B}_1(0) \subset \mathbb{R}^n$ in $n \geq 2$
- Cone over $S^2(1/\sqrt{3}) \times S^2(1/\sqrt{3})$
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Proposition

For the cone over circle of length $\alpha < 2\pi$

$$W\left(P_t\delta_x, P_t\delta_y\right) = \begin{cases} d(x,y) - \sqrt{\pi t} \cdot \frac{2}{\alpha} \sin \frac{\alpha}{2} + O(t), & \text{if } x \text{ or } y \text{ is the vertex} \\ d(x,y) + o(t), & \text{else.} \end{cases}$$

Gradient Flows on Time-dependent MM-Spaces

Definition

An absolutely continuous curve $(x_t)_{t\in J}$ will be called *dynamic backward* EVI^-*-gradient flow* for V if for all $t \in J$ and all $z \in Dom(V_t)$

$$\frac{1}{2}\partial_s^- d_{s,t}^2(x_s, z)\Big|_{s=t-} \geq V_t(x_t) - V_t(z)$$

where

$$d_{s,t}(x,y) := \inf\left\{\int_0^1 |\dot{\gamma}^a|^2_{s+a(t-s)} da\right\}^{1/2}$$

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$(x_t)_{t\in J}$ is called *dynamic backward EVI(K,N)-gradient flow* if $\forall z, \forall t$

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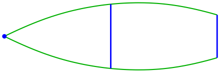
The Curvature-Dimension Condition $CD^*(K, N)$

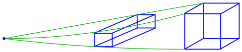
Def. A metric measure space (X, d, m) satisfies $CD^*(K, N)$

$\Longleftrightarrow$ S:= Ent(.) is (K,N)-convex on $(\mathcal{P}(X), W)$

$$\Longleftrightarrow \quad \partial_t S(\mu_t)\big|_{t=1} - \partial_t S(\mu_t)\big|_{t=0} \geq K \cdot W^2(\mu_0, \mu_1) + \frac{1}{N}\big(S(\mu_0) - S(\mu_1)\big)^2$$



sec ≥ 0 $\Longleftrightarrow$ dist concave



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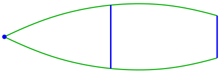

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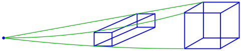

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Proposition. $CD^*(0,N) \iff \forall \mu_0,\mu_1 \in \mathcal{P}(X)$: $\exists$ geodesic $(\mu_t)_t$ s.t.

$$S_N(\mu_t|m) \quad \leq \quad (1-t) S_N(\mu_0|m) + t\, S_N(\mu_1|m))$$

where $S_N(\nu|m) = -\int_X \rho^{1-1/N}\, dm$ for $\nu = \rho \cdot m + \nu_s$.

Example: $S_N(\nu|m) = -m(A)^{1/N}$ if $\nu =$ unif. distrib. on $A \subset X$