Stability of heat kernel estimates for symmetric non-local Dirichlet forms and its applications

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Outline

1. **Motivation**
2. **Main results**
   - Heat kernel estimates
   - Harnack inequalities
3. **Long range random walks in random media**

Stability of heat kernel estimates: diffusion case

- Gaussian HKE: Let $L = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ be a uniform elliptic div. form on $\mathbb{R}^d$ ($\sigma^{-1}I \leqslant (a_{ij}(x)) \leqslant \sigma I$ for some $\sigma > 0$). Then,

$$\frac{c_1}{t^{d/2}} \exp\left(-c_2\frac{|x-y|^2}{t}\right) \leqslant p(t,x,y) \leqslant \frac{c_3}{t^{d/2}} \exp\left(-c_4\frac{|x-y|^2}{t}\right)$$

  for all $t > 0$ and $x, y \in \mathbb{R}^d$, see Aronson ('67).

- Stability of Gaussian HKE:
  Gaussian HKE $\Leftrightarrow$ $VD + PI(2) \Leftrightarrow PHI(2)$,
  see Grigor'yan ('91), Saloff-Coste ('92), Sturm ('96).

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Stability of heat kernel estimates: diffusion case

- Sub-Gaussian HKE: Diffusions on 'nice' fractals M: $\exists d_w \geqslant 2$ such that

$$\frac{c_1}{\mu(B(x,t^{1/d_w}))}\exp\left(-c_2\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) \leqslant p(t,x,y) \leqslant \frac{c_3}{\mu(B(x,t^{1/d_w}))}\exp\left(-c_4\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

for all $t>0$ and $x,y\in M$. See Barlow-Perkins, $\cdots$

- Stability of sub-Gaussian HKE:
Sub-Gaussian HKE $\Leftrightarrow$ $VD + PI(d_w)$+CS$(d_w)$$\Leftrightarrow$ $PHI(d_w)$.
See Barlow-Bass, Barlow-Bass-Kumagai, Andres-Barlow, Grigor'yan-Hu-Lau.

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Stability of heat kernel estimates: jump case

Theorem (Chen-Kumagai, '03)

Let $\mu(B(x,r)) \asymp r^d$ for all $x \in M$ and $r > 0$, and $\alpha \in (0,2)$. Then,

$$\mathcal{E}(f) \asymp \iint_{M\times M} \frac{(f(x)-f(y))^2}{d(x,y)^{d+\alpha}}\,\mu(dx)\,\mu(dy)$$

if and only if

$$p(t,x,y) \asymp \frac{1}{t^{d/\alpha}} \wedge \frac{t}{d(x,y)^{d+\alpha}}, \quad t>0, x,y \in M.$$

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Motivation

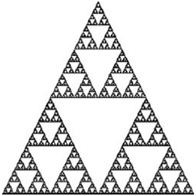
- On fractals, one can define α-stable process even for $2 \leqslant \alpha < d_w$.

Question

(Question from example) Let M be a Sierpinski gasket on $\mathbb{R}^2$, and $\mu(B(x,r)) \asymp r^d$ with $d = \frac{\log 3}{\log 2}$. Let

$$\mathcal{E}(f) \asymp \iint_{M\times M} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,\mu(dx)\,\mu(dy),$$

where $\alpha \in (0, \frac{\log 5}{\log 2})$ (possibly $\alpha \geqslant 2$). What is HKE?



Solution: Stability

- Brownian motions on $SG(\mathbb{R}^2)$:

$$p_*(t,x,y) \asymp t^{-d/d_w} \exp\left(-c\left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right),$$

where $d_w = \frac{\log 5}{\log 2} > 2$, see Barlow-Perkins.

- Subordination:

$$p_*^S(t,x,y) \asymp \frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}},$$

where $\alpha \in (0, d_w)$. Moreover,

$$\mathcal{E}_*^S(f) \asymp \iint_{M\times M} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,\mu(dx)\,\mu(dy).$$

- Stability:

$$\mathcal{E}(f) \asymp \iint_{M\times M} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,\mu(dx)\,\mu(dy)$$

implies

$$p(t,x,y) \asymp p_*^S(t,x,y).$$

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Settings

- MMS (M, d, μ). Let $V(x,r) = \mu(B(x,r))$ for all $x \in M$ and $r > 0$.
- VD and RVD

$$c_1\left(\frac{R}{r}\right)^{d_1} \leqslant \frac{V(x,R)}{V(x,r)} \leqslant c_2\left(\frac{R}{r}\right)^{d_2}, \quad x \in M, 0 < r < R.$$

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$$\mathcal{E}(f,g) = \iint_{M\times M} (f(x) - f(y))(g(x) - g(y))\, J(dx,dy).$$

- Scaling function ϕ:

$$c_3\left(\frac{R}{r}\right)^{\beta_1} \leqslant \frac{\phi(R)}{\phi(r)} \leqslant c_4\left(\frac{R}{r}\right)^{\beta_2}, \quad 0 < r < R.$$

- Example: BM $\phi(r) = r^2$; symmetric α-stable $\phi(r) = r^\alpha$.

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Heat kernel

- *HK*(ϕ):

$$p(t,x,y) \asymp \frac{1}{V(x,\phi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi(d(x,y))}.$$

- Example: symmetric α-stable processes on $\mathbb{R}^d$

$$p(t,x,y) \asymp \frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}$$

Jump kernel

- J_ϕ:

$$J(x,y) \asymp \frac{1}{V(x, d(x,y))\phi(d(x,y))}.$$

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Main result: heat kernel estimates

Theorem (Chen-Kumagai-W.)

The following are equivalent:

(i) $HK(\phi)$

(ii) J_ϕ *and* $CSJ(\phi)$.

- $CSJ(\phi)$: For $0 < r \leqslant R, f \in \mathcal{F}$ and almost all $x \in M$, there exists a cutoff function φ for $B(x,R) \subset B(x,R+r)$ so that the following holds:

$$\int_{U^*} f^2\, d\Gamma(\varphi,\varphi) \leqslant C_1 \int_{U\times U^*} (f(x)-f(y))^2 J(dx,dy) + \frac{C_2}{\phi(r)} \int_{U^*} f^2\, d\mu,$$

where $U^* = B(x, R+(1+C_0)r) \setminus B(x, R-C_0 r)$, $C_0 \in (0,1]$ and $U = B(x,R+r) \setminus B(x,R)$.

- $CSJ(\phi)$ holds trivially for the case that $\phi(r) = r^\alpha$ with $\alpha \in (0,2)$; for diffusions, we should take U^* as U (Andres-Barlow, '15).

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Counterexample

Example (J_ϕ only does not imply $HK(\phi)$.)

Let $M = \mathbb{R}^d$, $\phi(r) = r^\alpha + r^\beta$ with $0 < \alpha < 2 < \beta$, and

$$J(x,y) \asymp \frac{1}{|x-y|^d \phi(|x-y|)}, \quad x, y \in \mathbb{R}^d.$$

Then, J_ϕ holds, but $HK(\phi)$ does not hold.

- $CSJ(\phi^*)$ holds with $\phi^*(r) \asymp r^\alpha + r^2$.
- $$HK(\phi) \Longleftrightarrow J_\phi \ + \ CSJ(\phi).$$
- Stability of $HK(\alpha)$ for d-set also proved by Grigor'yan-E. Hu-J. Hu, Murugan-Saloff-Coste.

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Harmonic and parabolic functions

$X := (X_t)_{t\geqslant 0}$ is a MP, and $Z := (V_s, X_s)_{s\geqslant 0}$ is a time-space process where $V_s = V_0 - s$.

- $u(x)$ is *harmonic* on an open set $D \subset M$, if for every relatively compact open subset D_1 of D, $u(x) = \mathbb{E}^x u(X_{\tau_{D_1}})$ for all $x \in D_1$.

- $u(t,x)$ is *parabolic* on an open set $D \subset (0,\infty) \times M$, if for every relatively compact open subset D_1 of D,

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for every $(t,x) \in D_1$.

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Parabolic Harnack inequalities

- *PHI*(ϕ): there exist constants $0 < C_1 < C_2 < C_3 < C_4$, $C_5 > 1$ and $C_6 > 0$ such that for every $x_0 \in M$, $t_0 \geqslant 0$, $R > 0$ and for every non-negative function $u = u(t,x)$ that is parabolic on $Q := (t_0, t_0 + \phi(C_4R)) \times B(x_0, C_5R)$,

$$\sup_{Q_-} u \leqslant C_6 \inf_{Q_+} u,$$

where $Q_- := (t_0 + \phi(C_1R), t_0 + \phi(C_2R)) \times B(x_0, R)$ and $Q_+ := (t_0 + \phi(C_3R), t_0 + \phi(C_4R)) \times B(x_0, R)$.

Theorem (Grigor'yan; Saloff-Coste)

For diffusions on manifolds, Gaussian HKE $\Longleftrightarrow$ *PHI*(2) $\Longleftrightarrow$ $VD + PI(2)$.

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Main result: parabolic Harnack inequalities

Theorem (Chen-Kumagai-W.)

The following are equivalent:

(i) $PHI(\phi)$.

(ii) $J_{\phi,\leqslant}$, UJS, $CSJ(\phi)$ *and* $PI(\phi)$.

- $PI(\phi)$: There exist constants $C > 0$ and $\kappa \geqslant 1$ such that for any ball $B_r = B(x,r)$ and for any $f \in \mathcal{F}$,

$$\int_{B_r}\left(f-\frac{1}{\mu(B_r)}\int_{B_r} f\,d\mu\right)^2 d\mu \leqslant C\phi(r)\iint_{B_{\kappa r}\times B_{\kappa r}}(f(y)-f(x))^2 J(dx,dy).$$

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- $J_{\phi,\leqslant}$:

$$J(x,y) \leqslant \frac{c}{V(x,d(x,y))\phi(d(x,y))}.$$

- UJS: for almost all $x, y \in M$,

$$J(x,y) \leqslant \frac{c}{V(x,r)} \int_{B(x,r)} J(z,y)\,\mu(dz), \quad r \leqslant \frac{1}{2} d(x,y),$$

see Barlow-Bass-Kumagai.

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The following are equivalent:

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(iii) UJS, $UHK(\phi)$ and $NDL(\phi)$.

Corollary

$$HK(\phi) \Longleftrightarrow PHI(\phi) + J_{\phi,\geqslant}.$$

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Counterexample

Example (Dyda-Kassmann, '15; $PHI(\phi)$ only does not imply $HK(\phi)$.)

Let $M = \mathbb{R}^d$ and $0 < \alpha < 2$. For $0 < \theta < 1$ and $v \in \mathbb{R}^d$ with $|v| = 1$, define $A = \{h \in \mathbb{R}^d : |(h/|h|, v)| \geqslant \theta\}$ and

$$J(x, y) = 1_A(x - y)|x - y|^{-d-\alpha}.$$

Then, $PHI(\phi)$ holds, but $HK(\phi)$ does not hold.

Elliptic Harnack inequalities

Theorem (Chen-Kumagai-W.)

The following are equivalent:

- **(i)** $PHI(\phi)$.
- **(iv)** *EHI*, E_ϕ, *UJS and* $J_{\phi,\leqslant}$.
- **(v)** $WEHI(\phi)$, E_ϕ *and UJS.*
- **(vi)** *EHR*, E_ϕ *and UJS.*

- E_ϕ :

$$\mathbb{E}\tau_{B(x,r)} \asymp \phi(r), \quad x \in M, r > 0.$$

- $WEHI(\phi)$:

$$\frac{1}{\mu(B(x_0,r))}\int_{B(x_0,r)} u\,d\mu \leqslant c\left(\inf_{B(x_0,r)} u + \frac{\phi(r)}{\phi(R)} Tail\,(u_-; x_0, R)\right).$$

- $WEHI(\phi)$ implies *EHR*.

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$$\mathbb{E}\tau_{B(x,r)} \asymp \phi(r), \quad x \in M, r > 0.$$

- $WEHI(\phi)$:

$$\frac{1}{\mu(B(x_0,r))}\int_{B(x_0,r)} u\,d\mu \leqslant c\left(\inf_{B(x_0,r)} u + \frac{\phi(r)}{\phi(R)} Tail\,(u_-;x_0,R)\right).$$

- $WEHI(\phi)$ implies EHR.

- Z.-Q. Chen, T. Kumagai and Wang. Stability of heat kernel estimates for symmetric non-local Dirichlet forms. arXiv:1604.04035.
- Z.-Q. Chen, T. Kumagai and Wang. Stability of parabolic Harnack inequalities for symmetric non-local Dirichlet forms. arXiv:1609.07594.
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Long range random walks in random media

- Consider a countable set $\mathcal{V}$, and let $C : \mathcal{V} \times \mathcal{V} \to [0, \infty)$ such that $C_{x,y} = C_{y,x}$ and $0 < \sum_{y \in \mathcal{V}} C_{x,y} < \infty$ for any $x, y \in \mathcal{V}$.
- Long range random walks:

$$C_{x,y} = \frac{w_{x,y}}{|x-y|^{d+\alpha}}, \quad x, y \in \mathbb{Z}^d,$$

  where $\alpha \in (0, 2)$ and $\{w_{x,y} : x, y \in \mathbb{Z}^d\}$ is a sequence such that $w_{x,y} > 0$ for all $x \neq y$.
- Bass-Levin ('02): two-sided heat kernel estimates.
- $\{w_{x,,y} : x, y \in \mathbb{Z}^d\}$ is a sequence of random variables.

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QIP: definition

- For every $n \geqslant 1$ and $\omega \in \Omega$, we define a process $X^\omega_{\cdot,n}$ on $\mathbb{Z}^d_n := n^{-1}\mathbb{Z}^d$ by $X^\omega_{t,n} := n^{-1}X^\omega_{n^\alpha t}$ for any $t > 0$. Let $\mathbb{P}^\omega_{x,n}$ be the law of $X^\omega_{\cdot,n}$ with initial point $x \in \mathbb{Z}^d_n$.
- We say that the quenched invariance principle (QIP) holds for $X^\omega_\cdot$ with limit process being Y, if for $\mathbb{P}$-a.s. $\omega \in \Omega$ and every $T > 0$, $\mathbb{P}^\omega_{\cdot,n}$ converges weakly to $\mathbb{P}^Y_\cdot$ on the space of all probability measures on $\mathcal{D}([0,T];\mathbb{R}^d)$.

QIP: main result

Theorem (Chen-Kumagai-W.)

Let $d > 5 - 2\alpha$. Suppose that $\{w_{x,,y} : x, y \in \mathbb{Z}^d\}$ is a sequence of positive independent random variables such that

$$\sup_{x,y\in\mathbb{Z}^d} \mathbb{E}[w_{x,y}^{p}] + \sup_{x,y\in\mathbb{Z}^d} \mathbb{E}[w_{x,y}^{-q}] < \infty$$

with $p > (d+1)/(2-\alpha)$ and $q > 2(d+1)/d$. Then the quenched invariance principle holds for $X^{\omega}_{\cdot}$ with the limit process by a symmetric α-stable Lévy process Y on $\mathbb{R}^d$ with jumping measure $c_0|z|^{-d-\alpha}\,dz$, where $c_0 = \mathbb{E}w_{x,y}$.

- Reflected symmetric α-stable Lévy process Y on bounded domain.
- Symmetric Lévy processes Y with jumping measure $\frac{1}{|z|^d\phi(|z|)}\,dz$.

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The most common approach for QIP: corrector

- Harmonic embedding and the corrector:

$$X_t = M_t + \chi(\omega, X_t),$$

where M_t is a martingale and $\chi(\omega, X_t)$ is a corrector.

- However, for

$$C_{x,y} = \frac{w_{x,y}}{|x-y|^{d+\alpha}}, \quad x, y \in \mathbb{Z}^d$$

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$$\sum_{y \in \mathbb{Z}^d} |x-y|^2 C_{x,y} = \infty.$$

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A proper approach for QIP: long range random walks

- Discrete approximation of symmetric jump processes, see Chen-Kim-Kumagai ('13): Mosco convergence $\Longrightarrow$ convergence in L^2-sense.
- Tightness + Hölder regularity for harmonic function (two key ingredients), see Chen-Croydon-Kumagai ('15).

Proposition (Tightness)

Under some assumptions, for any $\varepsilon \in (0, 1/2)$ and for some $\theta \in (0,1)$, there is a constant $R_0 > 0$ such that for all $R > R_0$,

$$\sup_{x \in B(0,2R)} \mathbb{P}_x\left(\tau_{B(x,r)} \leqslant t\right) \leqslant C_1 \left(\frac{t}{r^\alpha}\right)^{1/2-\varepsilon}, \quad \forall\, t \geqslant r^{\theta\alpha},\ R^{\theta^2} \leqslant r \leqslant R.$$

- The proof is highly non-trivial!!!

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Proposition (Hölder regularity for parabolic functions)

Under some assumptions, there is a constant $R_0 > 0$ *such that for all* $R > R_0$, $x_0 \in B(0, R)$, $R^\theta \leqslant r \leqslant R$, $t_0 \geqslant 0$ *and parabolic function* q *on* $Q(t_0, x_0, 2r)$,

$$|q(s,x) - q(t,y)| \leqslant C_1 \|q\|_{\infty,r} \left(\frac{|t-s|^{1/\alpha} + |x-y|}{r} \right)^{\beta},$$

holds for $(s,x), (t,y) \in Q(t_0, x_0, r)$ *with* $(C_0^{-1}|s-t|)^{1/\alpha} + |x-y| \geqslant r^\theta$, *where*

$$\|q\|_{\infty,r} = \sup_{(s,x)\in[t_0, t_0+C_0(2r)^\alpha]\times\mathbb{Z}^d} q(s,x),$$

and $C_1 > 0$ *and* $\beta \in (0,1)$ *are constants independent of* R_0, x_0, t_0, R, r, s, t, x *and* y.

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Thank you for your attention!