Small Energy Ginzburg-Landau Minimizers in $\mathbb{R}^3$

Itai Shafrir

Technion—I.I.T.

April 2017

Based on a joint work with Etienne Sandier (Université Paris-Est)

$\mathbb{R}^2$-valued Ginzburg-Landau Minimizers on $\mathbb{R}^2$

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- A crucial estimate (**Sandier**):
$$u \text{ loc. min. } \implies E(u;B_R) \leq C \ln R \implies \int_{\mathbb{R}^2}(1-|u|^2)^2 < \infty.$$

Local minimizers among maps $u : \mathbb{R}^3 \to \mathbb{R}^2$

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- **Thm (Farina):** If $u : \mathbb{R}^N \to \mathbb{R}^2$ $(N = 3, 4)$ is a loc. min. with $\lim_{|x|\to\infty} |u(x)| = 1$, then u is a constant.

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"De Giorgi problem" for complex-valued maps

For which $N \geq 3$ a local minimizer $u : \mathbb{R}^N \to \mathbb{R}^2$ is necessarily **two dimensional**?

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(Use rescaling, $\varepsilon := \frac{1}{R}$, $u_\varepsilon(x) = u(Rx)$)

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Strategy of the proof of **Thm 2'**

- Assume $\big|1 - |u(0)|\big| > \lambda \overset{\diamond}{\Rightarrow} E(u; B_\alpha) \geq c(\lambda)$.
- Find $\tilde{R} \gg 1$ s.t. $E(u; B_{\tilde{R}}) = o(\tilde{R})$.
- By monotonicity: $c/\alpha \leq E(u; B_\alpha)/\alpha \leq E(u; B_{\tilde{R}})/\tilde{R} = o(1)$.

  Contradiction!

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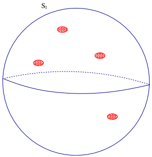
$\{|\tilde{u}_\varepsilon| < 7/8\} \subset \bigcup_{i=1}^k D_{r_i}(x_i)$, $\deg(\tilde{u}_\varepsilon / |\tilde{u}_\varepsilon|, \partial D_{r_i}) = d_i$.



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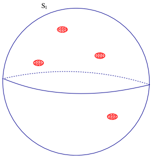

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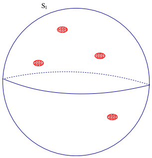

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But $\sum_{i=1}^k d_i = 0 \Rightarrow d_i = 0, \forall i$!!

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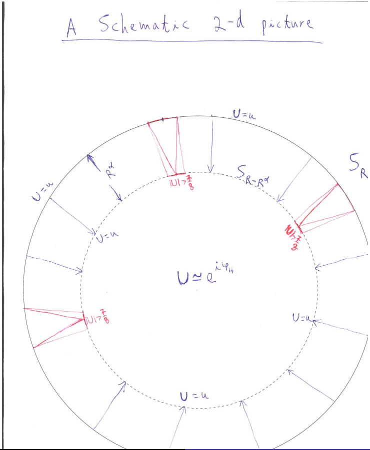

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- Iterate till you get η_k small enough and apply a known η-ellipticity result.

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where $W : \mathbb{R}^2 \to [0, \infty)$, $W = 0$ on Γ, a closed smooth curve.

Thank you for your attention!