Maximum principles at infinity on Riemannian manifolds and the Ahlfors-Khas'minskii duality

Joint works with

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- $(X, \langle\, , \rangle)$ Riemannian, $u \in C^2(X)$.
- Finite maximum principle: if $x_0 \in X$ max point of u,

$$u(x_0) = \sup_X u, \qquad |\nabla u(x_0)| = 0, \qquad \nabla^2 u(x_0) \le 0$$

- X noncompact, u bounded above. Look for $\{x_k\} \subset X$ satisfying:

$$(Ekeland) \qquad u(x_k) \to \sup_X u, \quad |\nabla u(x_k)| \to 0$$

$$(Omori) \qquad u(x_k) \to \sup_X u, \quad |\nabla u(x_k)| \to 0, \quad \nabla^2 u(x_k) \le \frac{1}{k}\langle\, , \rangle$$

$$(Yau) \qquad u(x_k) \to \sup_X u, \quad |\nabla u(x_k)| \to 0, \quad \Delta u(x_k) \le \frac{1}{k}$$

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- (X, d) metric. Then [Ekeland '74, Weston '77, Sullivan '81]

$$X \text{ complete} \iff X \text{ has the Ekeland principle}$$

Definition

X has the Ekeland principle if $\forall\, u \in USC(X)$ bounded above, there exists a sequence $\{x_k\} \subset X$ such that:

$$u(x_k) \to \sup_X u, \qquad u(y) \le u(x_k) + \frac{1}{k}\mathrm{dist}(x_k, y) \quad \forall\, y \in X.$$

- Omori (Yau) holds if X complete and, for $r(x) = \mathrm{dist}(x, o) \ge 1$

$$\mathrm{Sect} \ge -B^2 r^2 \qquad (\mathrm{Ric} \ge -B^2 r^2);$$

$X \to \mathbb{R}^n$ proper with bounded mean curvature.

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An Example [H. Omori '67, M.-Rigoli '10]

$\varphi : X^m \to \mathbb{R}^{2m-1}$ isometric immersion, X complete.

$v \in \mathbb{S}^{2m}$. Non-degenerate cone:

$$C_{v,\varepsilon} = \left\{ x \in \mathbb{R}^{2m+1} \quad : \quad \langle \frac{x}{|x|}, v \rangle \geq \varepsilon \right\}.$$

Theorem

If $-B^2 r^2 \leq \mathrm{Sect} \leq 0$, for a constant $B > 0$, then X cannot be contained into a non-degenerate cone of $\mathbb{R}^{2m-1}$.

Suppose $\varphi(X) \subset C_{v,\varepsilon}$. Fix $T = \langle \varphi(x_0), v \rangle$, $a \in (0, \varepsilon)$. Define

$$u(x) = \sqrt{T^2 + a^2 |\varphi(x)|^2} - \langle \varphi(x), v \rangle$$

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$x \in \{u > 0\}$, $W \in T_x X$, $|W| = 1$

$$\nabla^2 u(W, W) = \frac{a^2(1 + \langle \mathrm{II}(W, W), \varphi \rangle}{\sqrt{T^2 + a^2|\varphi|^2}} - \langle \mathrm{II}(W, W), v \rangle - \frac{a^4 \langle W, \varphi \rangle^2}{(T^2 + a^2|\varphi|^2)}$$

$$\geq \frac{a^2 T^2}{(T^2 + a^2|\varphi|^2)^{3/2}} + |\,\mathrm{II}(W, W)|\Big\{ \dots \Big\}.$$

[Otsuki]: $\exists\, W \,:\, \mathrm{II}(W, W) = 0$

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$$\exists\, w \in C^2(X\backslash K) \quad (K \text{ cpt.}) \text{ with}$$

$$w \le 0 \text{ on } X\backslash K, \qquad w(x) \to -\infty \text{ as } x \text{ diverges},$$

$$|\nabla w| \le 1, \qquad \nabla^2 w \ge -\langle\,,\rangle \qquad (\Delta w \ge -1), \quad \lambda > 0.$$

- [Pigola-Rigoli-Setti '05 + Kim-Lee, Fontenele-Barreto, Bessa-Lima-Pessoa]

- Weak maximum principles [Pigola-Rigoli-Setti]. Look for $\{x_k\} \subset X$ satisfying

$$\text{(weak Hessian)} \qquad u(x_k) \to \sup_X u, \qquad \nabla^2 u(x_k) \le \frac{1}{k}\langle\,,\rangle$$

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ADVANTAGES:

- Weak formulation: Weak Laplacian principle holds iff $\forall\, u \in C^2(X)$ bounded above and solving

$$\Delta u \geq f(u) \quad \text{on} \ \ \Omega_\gamma = \{u > \gamma\}$$

for some $f \in C(\mathbb{R})$, then $f(\sup_X u) \leq 0$.

- Volume growth criteria:

$$\liminf_{r\to+\infty} \frac{\log \mathrm{vol}(B_r)}{r^2} < +\infty \quad \Longrightarrow \quad \Delta \text{ has weak Laplacian principle.}$$

- Generalizations: for

$$0 < \mathcal{A} \in C(\mathbb{R}^+), \quad 0 < b \in C(X), \quad 0 < l \in C(\mathbb{R}^+), \quad f \in C(\mathbb{R})$$

Def: $(bl)^{-1}\Delta_{\mathcal{A}}$ has weak max. principle if

$$\mathrm{div}\Big(\mathcal{A}(|\nabla u|)\nabla u\Big) \geq b(x)f(u)l(|\nabla u|) \quad \text{on} \ \ \Omega_\gamma \quad \Rightarrow f(\sup_X u) \leq 0.$$

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RELATION WITH STOCHASTIC ANALYSIS

X Riemannian, $p(x, y, t)$ heat Kernel of X.

Brownian motion

$$\mathcal{B}_t : (\Omega, \mathcal{F}_t, \mathbb{P}) \to X$$

$\overline{X} = X \cup \{\infty\}$ Alexandrov compactification.

$$\mathcal{P}\big(\mathcal{B}_t \in X \;:\; \mathcal{B}_0 = x\big) \doteq \int_X p(x, y, t)\mathrm{d}y \leq 1.$$

If it is 1 for some (any) (x, t), we say that X is stochastically complete.

Y Martingale on X. If

$$\mathcal{P}\big(Y_t \in X \;:\; Y_0 = x\big) = 1 \qquad \text{for each } (x, t),$$

we say that X is martingale complete [M. Emery]

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$$\mathcal{P}\big(\mathcal{B}_t \in X \ : \ \mathcal{B}_0 = x\big) \doteq \int_X p(x, y, t) \mathrm{d}y \leq 1.$$

If it is 1 for some (any) (x, t), we say that X is stochastically complete.

Y Martingale on X. If

$$\mathcal{P}\big(Y_t \in X \ : \ Y_0 = x\big) = 1 \qquad \text{for each } (x, t),$$

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1) X has weak Laplacian principle;
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$$\exists\, w \in C^2(X \backslash K) \quad (K \text{ cpt.}) \text{ with}$$

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Krylov and Harvey-Lawson's approach

- $J^2(X) \to X$ 2-jet bundle, with fibers $J^2_x(X)$

$$\begin{aligned} J^2(X) &= \mathbf{R} \oplus T^*X \oplus \mathrm{Sym}^2(T^*X) \\ J &\mapsto (x, r, p, A) \end{aligned}$$

- For $u \in C^2(X)$, $J^2 u : X \to J^2(X)$,

$$J^2_x u = (u(x), \mathrm{d}u(x), \nabla^2 u(x))$$

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$$\Delta u \geq f(u) \iff J^2 u \in F \doteq \left\{ \mathrm{tr}(A) \geq f(r) \right\}$$

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$$\widetilde{F} = \neg\big(-\mathrm{Int}F\big) \qquad \sim \text{ is a duality: } \quad \widetilde{F \cap G} = \widetilde{F} \cup \widetilde{G}, \ \ \widetilde{\widetilde{F}} = F.$$

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F-Subharmonics and duality

- $u \in C^2(X)$ is F-subharmonic if $J^2_x u \in F_x \quad \forall x \in X$.
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Ahlfors and Khas'minskii property

X^m Riemannian, F subequation, $F_0 \doteq F \cup \{r \le 0\}$.

Definition

F has the Ahlfors property iff for each $U \subset X$ open and $u \in F_0(\overline{U})$ bounded above,

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[Berestycki - Hamel - Rossi '07]

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$$K \subset X \text{ cpt.}, \quad h \in C(X \backslash K), \quad h < 0, \quad \begin{array}{l} h(x) \to -\infty \\ \text{as } x \to \infty \end{array}$$

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Let $F \subset J^2(X)$ subequation such that:

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Immersions and submersions

(1) $\sigma : X^m \to Y^n$ isometric immersion, proper, $\| \mathrm{II} \|_\infty < +\infty$.

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Then,

$$F_{n-k} \cup \widetilde{E} \text{ is Ahlfors on } Y \quad \Longleftrightarrow F_{m-k} \cup \widetilde{E} \text{ is Ahlfors on } X$$

(2) $\pi : X^m \to Y^n$ Riemannian submersion, compact fibers $X_y = \pi^{-1}\{y\}$.

II_y second fund. form of X_y, $\mathcal{A}$ integrability tensor.

Suppose $\|\mathcal{A}\|_\infty + \| \mathrm{II}_y \|_\infty \leq C$ for each y. Then

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Immersions and submersions

(1) $\sigma : X^m \to Y^n$ isometric immersion, proper, $\| \mathrm{II} \|_\infty < +\infty$.

For $k \leq m$, consider $F_k = \{\lambda_k(A) \geq f(r)\}$

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II_y second fund. form of X_y, $\mathcal{A}$ integrability tensor.

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