A class of highly degenerate elliptic operators: maximum principle and unusual phenomena

Based on a joint work with Isabeau Birindelli and Hitoshi Ishii

Giulio Galise

Sapienza Università di Roma

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$$F[u] := F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^N$$

$F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \mapsto \mathbb{R}$ continuous and degenerate elliptic, i.e.

$$F(\cdot,\cdot,\cdot,X) \leq F(\cdot,\cdot,\cdot,Y) \;\text{ whenever } X \leq Y \text{ in } \mathbb{S}^N,$$

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Maximum Principle

▷ **Weak Maximum Principle**

$$F[u] \geq 0 \text{ in } \Omega, \quad \limsup_{x \to \partial\Omega} u(x) \leq 0 \implies u \leq 0 \text{ in } \Omega$$

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$$F[u] \geq 0 \text{ in } \Omega, \quad u \leq 0 \text{ in } \Omega \implies \text{either } u < 0 \text{ or } u \equiv 0$$

A class of degenerate operators

For $X \in \mathbb{S}^N$ let $\lambda_1(X) \leq \ldots \leq \lambda_N(X) \in \text{spec}(X)$ and $N \geq k \in \mathbb{N}$.
We shall consider

$$\mathcal{P}_k^-(X) = \lambda_1(X) + \ldots + \lambda_k(X)$$
$$\mathcal{P}_k^+(X) = \lambda_{N-k+1}(X) + \ldots + \lambda_N(X) \qquad ,$$

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▷ *Handlebodies and p-convexity* [Sha, J. Differential Geom. 1987]

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▷ *Dirichlet Duality and the Nonlinear Dirichlet Problem* [Harvey-Lawson, Comm. Pure Appl. Math. 2009]

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$\mathcal{P}_k^-(\cdot)$ is a degenerate elliptic operator

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If $k < N$, it is furthermore *degenerate in any direction* $v \in \mathbb{R}^N$, i.e.

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just take $X = 0$ and use $\mathrm{spec}(v \otimes v) = \{0, \dots, 0, 1\}$

Consider

$$\begin{cases} \mathcal{P}_k^-(D^2u) + H(x, Du) + \mu u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{DP}$$

where $f \in C(\Omega)$, $\mu \in \mathbb{R}$, the Hamiltonian $H \in C(\Omega \times \mathbb{R}^N)$ and

$$|H(x,\xi)| \leq b\,|\xi| \qquad \forall (x,\xi) \in \Omega \times \mathbb{R}^N \tag{SC 1}$$

e.g. $H(x, Du) = b(x)|Du|$ or $H(x, Du) = \langle b(x), Du \rangle$ with $b \in L^\infty$

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Aims

- Validity (lack of) of the maximum/minimum principle, be it weak or strong (w.l.o.g. $\mu \geq 0$)
- Regularity of the solutions of (DP)
- Existence of principal eigenvalues and eigenfunctions
- Point out differences with respect to the uniformly elliptic case

Strong minimum principle

The strong minimum principle is closely related to the *Hopf lemma* and the *weak Harnack inequality*

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$$\lambda_i(D^2 w) = \frac{w'(|x|)}{|x|} = -2\gamma\left(1-|x|^2\right)^{\gamma-1} \qquad \text{for } i = 1, \dots, N-1$$

$$\lambda_N(D^2 w) = w''(|x|) = \underbrace{-2\gamma\left(1-|x|^2\right)^{\gamma-1}}_{=\lambda_i(D^2 w)} + 4|x|^2\gamma(\gamma-1)\left(1-|x|^2\right)^{\gamma-2}$$

$$\begin{cases} \mathcal{P}_k^-(D^2 w) < 0 & \text{in } B_1 \\ w > 0 & \text{in } B_1 \\ w = \partial_\nu w = 0 & \text{on } \partial B_1 \end{cases}$$

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Then Hopf lemma **does not hold** for $\mathcal{P}_k^-$ if $k < N$:

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Likewise for the weak Harnack inequality

Let

$$u(x_1,\dots,x_N)=\frac{1}{2}x_N^2$$

then

$$D^2u=\mathrm{diag}[0,\dots,0,1]$$

and ($k<N$)

$$\mathcal{P}_k^-(D^2u)=0 \quad \text{in } B_2$$

Nevertheless for any $p>0$ and any $C>0$

$$\left(\frac{1}{|B_1|}\int_{B_1}u^p\right)^{\frac{1}{p}}\not\leq 0=C\inf_{B_1}u$$

Maximum and Minimum Principle

Under the assumption (SC 1) and for any $k < N$, the operator

$$\mathcal{P}_k^-(D^2\cdot) + H(x, D\cdot)$$

does not satisfy the strong minimum principle in any bounded domain Ω.

On the other hand the **weak minimum principle holds true** in

$$\Omega \subseteq B_R \quad \text{if} \quad bR \leq k$$

and the condition $bR \leq k$ is **sharp** (remember $H(x, Du) \approx b|Du|$).

The **strong maximum principle holds true** in any bounded domain since the boundary Hopf lemma applies to negative solutions u of

$$\mathcal{P}_k^-(D^2u) + H(x, Du) \geq 0 \quad \text{in } \Omega$$

Generalized principal eigenvalues

What about

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Linear uniformly elliptic case [**Berestycki-Nirenberg-Varadhan**, Comm. Pure Appl. Math. 1994]

$$F[u] := \mathrm{Tr}(A(x)D^2u) + b(x)\cdot Du + c(x)u$$

the validity of the weak maximum (minimum) principle is related to the positivity of the principal eigenvalue

$$\mu_1^+ := \sup\left\{\mu \in \mathbb{R}\colon\ \exists w \in W^{2,N}_{\mathrm{loc}}(\Omega),\ w > 0 \text{ and } F[u] + \mu w \le 0 \text{ in } \Omega\right\}$$

The BNV approach has been addressed in the fully nonlinear uniformly elliptic framework

[Busca-Esteban-Quaas, Birindelli-Demengel, Ishii-Yoshimura, Quaas-Sirakov, Armstrong, Patrizi, Ikoma-Ishii...]

$$F[u] := F(x, u, Du, D^2u) \quad \text{homogeneous of degree 1}$$

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(ii) $|\Omega| \to 0 \ \Rightarrow\ \mu_1^\pm \to +\infty$

(iii) $\mu < \mu_1^+ \ \Rightarrow\ F[\cdot] + \mu \cdot$ satisfies the **weak maximum principle** in Ω

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(iv) μ_1^+ and μ_1^- correspond respectively to a positive and negative principal eigenfunction

$$\mathcal{P}_k^-(D^2\cdot) + H(x, D\cdot) + \mu\cdot$$

$$H(x, t\,\xi) = t\,H(x,\xi) \qquad t > 0 \tag{SC 2}$$

$$|H(x,\xi) - H(y,\xi)| \le \omega\left(|x-y|\,(1+|\xi|)\right) \tag{SC 3}$$

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$$\mu_k^- = \sup\{\mu \in \mathbb{R} : \exists w < 0 \text{ in } \Omega,\ \mathcal{P}_k^-(D^2 w) + H(x, Dw) + \mu w \geq 0\}$$

and

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$$\overline{\boldsymbol{\mu}}_k^- = \sup\{\mu \in \mathbb{R} : \exists w < 0 \text{ in } \overline{\Omega},\ \mathcal{P}_k^-(D^2w) + H(x, Dw) + \mu w \ge 0\}$$

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$\overline{\boldsymbol{\mu}}_k^\pm = \boldsymbol{\mu}_k^\pm$?

The equality $\overline{\boldsymbol{\mu}}^\pm = \boldsymbol{\mu}^\pm$ holds for uniformly elliptic operators, while examples of degenerate (first order) operators s.t. $\overline{\boldsymbol{\mu}}^\pm < \boldsymbol{\mu}^\pm$ are exhibit in [Berestycki-Capuzzo Dolcetta-Porretta-Rossi, 2015].

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Under the assumptions (SC 2)-(SC 3), then the operator

$$\mathcal{P}_k^-(D^2\cdot) + H(x, D\cdot) + \mu\cdot$$

satisfies:

i) the **weak minimum principle** for $\mu < \overline{\mu}_k^-$

ii) the **weak maximum principle** for $\mu < \overline{\mu}_k^+$.

...To reach the values μ_k^- and μ_k^+ (the standard thresholds in the uniformly elliptic case) we shall need some further conditions!

The case $\overline{\boldsymbol{\mu}}_k^+$

$$\Omega \subseteq B_R \quad \text{and} \quad bR < k$$

Let

$$w(|x|) = \left(R^2 - |x|^2\right)^\gamma > 0 \quad \text{in } \overline{\Omega}$$

Then for any $\mu > 0$

$$\mathcal{P}_k^-(D^2 w) + H(x, Dw) + \mu w \leq 0 \qquad \text{for } \gamma = \gamma(\mu, b, k, R) \text{ big enough}$$

$$\boxed{\overline{\boldsymbol{\mu}}_k^+ = +\infty}$$

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The case $\overline{\mu}_k^+$

$$\Omega \subseteq B_R \quad \text{and} \quad bR < k$$

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$\mathcal{P}_k^-$ vs Δ

Maximum principle holds true for

$$\Delta \cdot + \mu \cdot = \lambda_1(D^2 \cdot) + \ldots + \lambda_N(D^2 \cdot) + \mu \cdot \ \text{ in } \Omega$$

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$$\mathcal{P}_{N-1}^-(D^2 \cdot) + \mu \cdot = \lambda_1(D^2 \cdot) + \ldots + \lambda_{N-1}(D^2 \cdot) + \cancel{\lambda_N(D^2 \cdot)} + \mu \cdot$$

satisfies the maximum principle for any $\mu \in \mathbb{R}$.

Instability of $\overline{\mu}_k^+$

Consider

$$\mathcal{P}_k^-(D^2\cdot) + \frac{k}{R}|D\cdot| \quad \text{in } \Omega_n = B_{R-\frac{1}{n}}$$

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On the other hand $w(|x|)=\left(R^2-|x|^2\right)^\gamma$ satisfies

$$\begin{cases} \mathcal{P}_k^-(D^2w)+\frac{k}{R}|Dw|+\frac{2\gamma k}{R^2}w\geq 0 & \text{in } \Omega=\cup_{n\in\mathbb{N}}\Omega_n \\ w=0 & \text{on } \partial\Omega \\ w>0 & \text{in } \Omega. \end{cases}$$

Hence this contradicts the maximum principle and

$$\overline{\mu}_k^+(\Omega)\leq\frac{2\gamma k}{R^2}$$

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Let $R_1 \leq 1$ s.t. $B_{R_1} \subseteq \Omega$. No blow-up phenomena, being

$$\overline{\boldsymbol{\mu}}_k^- \leq \frac{C(b, k)}{R_1^2}$$

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Ω_n: domains whose measure goes to zero but whose principal eigenvalue stays equal to zero !!!

...Unusual phenomena again: 2D-narrow domains shrinking to a line with bounded principal eigenvalue

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$$w(x_1, x_2) = -\sin nx_1 - \sin x_2$$

and $(x_1, x_2) \in \Omega_n := \left\{0 \leq \frac{nx_1+x_2}{2} \leq \pi,\ -\frac{\pi}{2} \leq \frac{nx_1-x_2}{2} \leq \frac{\pi}{2}\right\}$

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Ω_n narrow domains in the x_1-direction s.t.

$$\operatorname{diam}(\Omega_n) = 2\pi \quad \text{and} \quad \Omega_n \to \{0\} \times \left[-\frac{\pi}{2}, \frac{3}{2}\pi\right]$$

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Nevertheless

$$\begin{cases} \mathcal{P}_1^-(D^2 w) + 1 w \leq 0 & \text{in } \Omega_n \\ w = 0 & \text{on } \partial\Omega_n \\ w < 0 & \text{in } \Omega_n \end{cases}$$

violating the minimum principle, hence

$$\overline{\boldsymbol{\mu}}_1^-(\Omega_n) \leq \mathbf{1} \quad \forall n \in \mathbb{N}$$

On the equivalence $\overline{\mu}_k^- = \mu_k^-$

$$\begin{cases} \mathcal{P}_k^-(D^2v) + H(x, Dv) + \mu v \le 0 & \text{in } \Omega \\ \liminf\limits_{x \to \partial\Omega} v \ge 0 \end{cases}$$

Minumum principle OK if $\mu < \overline{\mu}_k^- (\le \mu_k^-)$

How reach the value μ_k^- ?

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Warning : "$\partial\Omega$ flat", e.g. $v(x) = -x_N^{\gamma \in (0,1)}$ is a solution of

$$\mathcal{P}_k^-(D^2v) = 0 \quad \text{in } \mathbb{R}_+^N := \{x : x_N > 0\}$$

$$v = 0 \quad \text{on } \partial\mathbb{R}_+^N$$

$$\lim_{x \to \partial\mathbb{R}_+^N} \frac{v(x)}{d(x)} = -\lim_{x \to \partial\mathbb{R}_+^N} \frac{x_N^\gamma}{x_N} = -\infty$$

Convexity of Ω is needed...

Hula hoop domains

We shall consider a class $\mathcal{C}_R$ of convex domains Ω satisfying the following assumption: there exist $R > 0$ and $Y \subseteq \mathbb{R}^N$, depending on Ω, such that

$$\boxed{\Omega = \bigcap_{y \in Y} B_R(y)}$$

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Proposition

Let Ω be a bounded domain with C^2-boundary. Let $\kappa_i(x)$ denote the principal curvatures of $\partial\Omega$ at x for $i = 1, \dots, N-1$, set

$$\underline{\kappa} = \min\{\kappa_i(x) \, : \, i = 1, \dots, N-1, \; x \in \partial\Omega\},$$

and assume that $\underline{\kappa} > 0$. If $R \geq \frac{1}{\underline{\kappa}}$, then $\Omega \in \mathcal{C}_R$.

Theorem

Let $\Omega \in \mathcal{C}_R$. If H satisfies (SC 2)-(SC 3) and $bR < k$, then

$$\boldsymbol{\mu}_k^- = \overline{\boldsymbol{\mu}}_k^-$$

and the minimum principle holds true in Ω for

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Lipschitz regularity $(k = 1)$

$$\begin{cases} \mathcal{P}_1^-(D^2u) + H(x, Du) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{DP$_1$}$$

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(iv) u solution of (DP_1) ? At least for Ω unbounded, global regularity does not hold: $u(x) = x_N^{\gamma<\alpha} \notin C^{0,\alpha}(\overline{\mathbb{R}}_+^N)$, but it is a solution of

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...Again "hula hoop" condition

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$$u(x) \quad v_y(x) := u(y) + L\left(|x-y| - |x-y|^\theta\right) \qquad \text{in } B_\delta(y) \cap \Omega$$

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Eigenfunction for $\mathcal{P}_1^-$

Theorem

Let $\Omega \in \mathcal{C}_R$. If H satisfies (SC 2)-(SC 3) and $bR < 1$, then there exists a negative function $\psi_1 \in \mathrm{Lip}(\overline{\Omega})$ such that

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Step 3. *Rescaling* $v_n = \frac{u_n}{\|u_n\|_\infty} \to \psi_1 \in \text{Lip}(\overline{\Omega})$ and passing to the limit

$$\mathcal{P}_1^-(D^2\psi_1) + H(x, D\psi_1) + \mu_1^-\psi_1 = 0 \text{ in } \Omega, \quad \psi_1 = 0 \text{ on } \partial\Omega$$

Eigenfunction for $\mathcal{P}_1^-$

Theorem

Let $\Omega \in \mathcal{C}_R$. If H satisfies (SC 2)-(SC 3) and $bR < 1$, then there exists a negative function $\psi_1 \in \mathrm{Lip}(\overline{\Omega})$ such that

$$\begin{cases} \mathcal{P}_1^-(D^2\psi_1) + H(x, D\psi_1) + \boldsymbol{\mu}_1^-\psi_1 = 0 & \text{in } \Omega \\ \psi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Step 1. Let $\boldsymbol{\mu}_n \nearrow \boldsymbol{\mu}_1^-$. Then $\exists u_n \in \mathrm{Lip}(\overline{\Omega})$ negative solutions of

$$\mathcal{P}_1^-(D^2u_n) + H(x, Du_n) + \boldsymbol{\mu}_n u_n = 1 \text{ in } \Omega, \quad u_n = 0 \text{ on } \partial\Omega$$

Step 2. *Compactness* yields $\|u_n\|_\infty \to \infty$

Step 3. *Rescaling* $v_n = \frac{u_n}{\|u_n\|_\infty} \to \psi_1 \in \mathrm{Lip}(\overline{\Omega})$ and passing to the limit

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Step 4. *Strong maximum principle* yields $\psi_1 < 0$ in Ω.

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and

$$\mathcal{P}_1^-(D^2\psi_1) + \left(\frac{\pi}{2R}\right)^2 \psi_1 = 0 \text{ in } \Omega, \quad \psi_1 = 0 \text{ on } \partial\Omega$$

By definition $\boldsymbol{\mu}_1^- \geq \left(\frac{\pi}{2R}\right)^2$, on the other hand ψ_1 violates the minimum principle, hence $\boldsymbol{\mu}_1^- \leq \left(\frac{\pi}{2R}\right)^2$

$$\boxed{\boldsymbol{\mu}_1^- = \left(\frac{\pi}{2R}\right)^2}$$

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Thank you for your attention!!!