Formality notions for spaces and groups

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COMMUTATIVE DIFFERENTIAL GRADED ALGEBRAS

- Let $A = (A^\bullet, \mathrm{d})$ be a commutative, differential graded algebra over a field $\Bbbk$ of characteristic 0.
  - $A = \bigoplus_{i \geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - The multiplication $\cdot\colon A^i \otimes A^j \to A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|} ba$ for all homogeneous a and b.
  - The differential $\mathrm{d}\colon A^i \to A^{i+1}$ satisfies the graded Leibniz rule, i.e., $\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\,\mathrm{d}(b)$.
- The cohomology $H^\bullet(A)$ of the cochain complex (A, d) inherits an algebra structure from A.
- A cdga morphism $\varphi\colon A \to B$ is both an algebra map and a cochain map. Hence, φ induces a morphism $\varphi^*\colon H^\bullet(A) \to H^\bullet(B)$.
- The map φ is a quasi-isomorphism if φ^* is an isomorphism. Likewise, φ is a q-quasi-isomorphism (for some $q \geqslant 1$) if φ^* is an isomorphism in degrees $\leqslant q$ and is injective in degree $q+1$.

FORMALITY OF CDGAS

- Two cdgas, A and B, are *weakly (q-)equivalent* ($\simeq_q$) if there is a zig-zag of (q-)quasi-isomorphisms connecting A to B.
- (Sullivan 1977) A cdga (A, d) is *formal* (or just *q-formal*) if it is (q-)weakly equivalent to $(H^\bullet(A), d = 0)$.
- Formality implies uniform vanishing of all Massey products.
- E.g., if A is 1-formal, then all triple Massey products in $H^2(A)$ must vanish modulo indeterminancy: if $a, b, c \in H^1(A)$, and $ab = bc = 0$, then $\langle a, b, c\rangle = 0$ in $H^\bullet(A)/(a, c)$.
- (Halperin–Stasheff 1979) Let $\mathbb{K}/\Bbbk$ be a field extension. A $\Bbbk$-cdga (A, d) with $H^\bullet(A)$ of finite-type is formal if and only if the $\mathbb{K}$-cdga $(A \otimes \mathbb{K}, d \otimes \mathrm{id}_{\mathbb{K}})$ is formal.
- (S.–He Wang 2015) Suppose $\dim H^{\leqslant q+1}(A) < \infty$ and $H^0(A) = \Bbbk$. Then (A, d) is q-formal iff $(A \otimes \mathbb{K}, d \otimes \mathrm{id}_{\mathbb{K}})$ is q-formal.

ALGEBRAIC MODELS FOR SPACES

- To a large extent, the rational homotopy type of a space can be reconstructed from algebraic models associated to it.
- If the space is a smooth manifold M, the standard $\mathbb{R}$-model is the de Rham algebra $\Omega_{\mathrm{dR}}(M)$.
- More generally, any (path-connected) space X has an associated Sullivan $\mathbb{Q}$-cdga, $A_{\mathrm{PL}}(X)$. In particular, $H^\bullet(A_{\mathrm{PL}}(X)) = H^\bullet(X, \mathbb{Q})$.
- An *algebraic (q-)model* (over $\Bbbk$) for X is a $\Bbbk$-cgda (A, d) which is (q-) weakly equivalent to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$.
- For instance, every smooth, quasi-projective variety X admits a finite-dimensional, rational model $A = A(\overline{X}, D)$, constructed by Morgan from a normal-crossings compactification $\overline{X} = X \cup D$.

FORMALITY OF SPACES

- A space X is *(q-)formal* if $A_{\mathrm{PL}}(X)$ has this property, i.e., $(H^{\bullet}(X,\mathbb{Q}), d=0)$ is a (q-)model for X.
- Spheres, Lie groups and their classifying spaces, homogeneous spaces G/K with $\mathrm{rk}G = \mathrm{rk}K$, and $K(\pi,n)$'s with $n \geqslant 2$ are formal.
- Formality is preserved under (finite) direct products and wedges of spaces, as well as connected sums of manifolds.
- The 1-formality property of X depends only on $\pi_1(X)$.
- (Macinic 2010) If X is a q-formal CW-complex of dimension at most $q+1$, then X is formal.
- A *Koszul algebra* is a graded $\Bbbk$-algebra such that $\mathrm{Tor}^{A}_{s}(\Bbbk,\Bbbk)_t = 0$ for all $s \neq t$.
- (Papadima–Yuzvinsky 1999) Suppose $H^{\bullet}(X,\Bbbk)$ is a Koszul algebra. Then X is formal if and only if X is 1-formal.

GEOMETRY AND FORMALITY

- (Stasheff 1983) Let X be a k-connected CW-complex of dimension n. If $n \leqslant 3k + 1$, then X is formal.
- (Miller 1979) If M is a closed, k-connected manifold of dimension $n \leqslant 4k + 2$, then M is formal. In particular, all simply-connected, closed manifolds of dimension at most 6 are formal.
- (Fernández–Muñoz 2004) There exist closed, simply-connected, non-formal manifolds of dimension 7.
- (Deligne–Griffiths–Morgan–Sullivan 1975) All compact Kähler manifolds are formal.
- (Papadima–S. 2015) If M is a compact Sasakian manifold of dimension $2n + 1$, then M is $(2n - 1)$-formal.

PURITY IMPLIES FORMALITY

- (Morgan 1978) Let X be a smooth, quasi-projective variety. If $W_1 H^1(X, \mathbb{C}) = 0$, then X is 1-formal.
- (Dupont 2016) More generally, suppose either
  - $H^k(X)$ is pure of weight k, for all $k \leqslant q+1$, or
  - $H^k(X)$ is pure of weight $2k$, for all $k \leqslant q$.

  Then X is q-formal.
- In particular, complements of hypersurfaces in $\mathbb{CP}^n$ are 1-formal. Thus, complements of plane algebraic curves are formal.
- Complements of linear and toric arrangements are formal, but complements of elliptic arrangements may be non-1-formal.

RESONANCE VARIETIES OF A CDGA

- Assume the cdga (A, d) is connected, i.e., $A^0 = \Bbbk$, and of finite-type, i.e., $\dim A^i < \infty$ for all $i \geqslant 0$.
- For each $a \in Z^1(A) \cong H^1(A)$, we have a cochain complex,

$$(A^\bullet, \delta_a)\colon\ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u + \mathrm{d}\, u$, for all $u \in A^i$.
- The *resonance varieties* of (A, d) are the sets

$$\mathcal{R}^i(A) = \{a \in H^1(A) \mid H^i(A^\bullet, \delta_a) \neq 0\}.$$

- An element $a \in H^1(A)$ belongs to $\mathcal{R}^i(A)$ if and only if

$$\operatorname{rank} \delta_a^{i+1} + \operatorname{rank} \delta_a^i < b_i(A).$$

- If $\mathrm{d} = 0$, then the resonance varieties of A are homogeneous.

Cohomology jump loci of spaces

- The *resonance varieties* of a connected, finite-type CW-complex X are the subsets $\mathcal{R}^i(X) := \mathcal{R}^i(H^\bullet(X,\mathbb{C}), d=0)$ of $H^1(X,\mathbb{C})$.
- The variety $\mathcal{R}^1(X)$ depends only on the group $G=\pi_1(X)$; in fact, only on the second nilpotent quotient $G/\gamma_3(G)$.
- The *characteristic varieties* of X are the Zariski closed sets of the character group of G given by

$$\mathcal{V}^i(X) = \{\rho \in \operatorname{Hom}(G,\mathbb{C}^*) \mid H^i(X,\mathbb{C}_\rho) \neq 0\}.$$

- The variety $\mathcal{V}^1(X)$ depends only on the group $G=\pi_1(X)$; in fact, only on the second derived quotient G/G''.
- Given any subvariety $W \subset (\mathbb{C}^*)^n$, there is a finitely presented group G such that $G_{\mathrm{ab}} = \mathbb{Z}^n$ and $\mathcal{V}^1(G) = W$.

THE TANGENT CONE THEOREM

- (Libgober 2002, Dimca–Papadima–S. 2009)

$$\tau_1(\mathcal{V}^i(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i(X)) \subseteq \mathcal{R}^i(X).$$

- Here, if $W \subset (\mathbb{C}^*)^n$ is an algebraic subset, then

$$\tau_1(W) := \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \text{ for all } \lambda \in \mathbb{C}\}$$

  is a finite union of rationally defined linear subspaces of $\mathbb{C}^n$.

- (DPS 2009/DP 2014) If X is q-formal, then, for all $i \leqslant q$,

$$\tau_1(\mathcal{V}^i(X)) = \mathrm{TC}_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X).$$

- This theorem yields a very efficient formality test.

Example

Let $G = \langle x_1, x_2 \mid [x_1, [x_1, x_2]]\rangle$. Then $\mathcal{V}^1(\pi) = \{t_1 = 1\}$, and so $\mathrm{TC}_1(\mathcal{V}^1(\pi)) = \{x_1 = 0\}$. But $\mathcal{R}^1(\pi) = \mathbb{C}^2$, and so π is not 1-formal.

Example

Let $G = \langle x_1, \dots, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4]\rangle$. Then $\mathcal{R}^1(\pi) = \{z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0\}$: a quadric which splits into two linear subspaces over $\mathbb{R}$, but is irreducible over $\mathbb{Q}$. Thus, π is not 1-formal.

Example

Let $\mathrm{Conf}_n(E)$ be the configuration space of n labeled points of an elliptic curve. Then

$$\mathcal{R}^1(\mathrm{Conf}_n(E)) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \leqslant i < j < n \end{array} \right\}.$$

For $n \geqslant 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, $\mathrm{Conf}_n(E)$ is not 1-formal.

Associated graded Lie algebras

- The *lower central series* of a group G is defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$.
- This forms a filtration of G by characteristic subgroups. The LCS quotients, $\gamma_k G/\gamma_{k+1} G$, are abelian groups.
- The group commutator induces a graded Lie algebra structure on

$$\operatorname{gr}(G, \Bbbk) = \bigoplus_{k \geqslant 1} (\gamma_k G/\gamma_{k+1} G) \otimes_{\mathbb{Z}} \Bbbk.$$

- Assume G is finitely generated. Then $\operatorname{gr}(G, \Bbbk)$ is also finitely generated (in degree 1) by $\operatorname{gr}_1(G, \Bbbk) = H_1(G, \Bbbk)$.
- For instance, if F_n is the free group of rank n, then $\operatorname{gr}(F_n; \Bbbk)$ is the free graded Lie algebra $\operatorname{Lie}(\Bbbk^n)$.

HOLONOMY LIE ALGEBRAS

- Let A be a commutative graded algebra with $A^0 = \Bbbk$ and $\dim A^1 < \infty$. Set $A_i = (A^i)^*$.
- The multiplication map $A^1 \otimes_{\Bbbk} A^1 \to A^2$ factors through a linear map $\mu_A \colon A^1 \wedge A^1 \to A^2$.
- Dualizing, and identifying $(A^1 \wedge A^1)^* \cong A_1 \wedge A_1$, we obtain a linear map, $\mu_A^* \colon A_2 \to A_1 \wedge A_1 \cong \mathrm{Lie}_2(A_1)$.
- The *holonomy Lie algebra* of A is the quotient

$$\mathfrak{h}(A) = \mathrm{Lie}(A_1)/\langle \mathrm{im}\, \mu_A^* \rangle.$$

- $\mathfrak{h}(A)$ is a quadratic Lie algebra, which depends only on the quadratic closure, $\bar{A} := \bigwedge(A^1)/\langle \ker \mu_A \rangle$. In fact, $U(\mathfrak{h}(A)) = \bar{A}^{!}$.
- For a f.g. group G, set $\mathfrak{h}(G, \Bbbk) := \mathfrak{h}(H^{\bullet}(G, \Bbbk))$. There is then a canonical surjection $\mathfrak{h}(G, \Bbbk) \twoheadrightarrow \mathrm{gr}(G, \Bbbk)$, which is an isomorphism precisely when $\mathrm{gr}(G, \Bbbk)$ is quadratic.

MALCEV LIE ALGEBRAS

- Let G be a f.g. group. The successive quotients of G by the terms of the LCS form a tower of finitely generated, nilpotent groups,

$$\cdots \longrightarrow G/\gamma_4 G \longrightarrow G/\gamma_3 G \longrightarrow G/\gamma_2 G = G_{\mathrm{ab}}\ .$$

- (Malcev 1951) It is possible to replace each nilpotent quotient N_k by $N_k \otimes \Bbbk$, the (rationally defined) nilpotent Lie group associated to the discrete, torsion-free nilpotent group $N_k/\mathrm{tors}(N_k)$.
- The inverse limit, $\mathfrak{M}(G;\Bbbk) = \varprojlim_k (G/\gamma_k G) \otimes \Bbbk$, is a prounipotent, filtered Lie group, called the *prounipotent completion* of G over $\Bbbk$.
- The pronilpotent Lie algebra

$$\mathfrak{m}(G;\Bbbk) := \varprojlim_k \mathfrak{Lie}((G/\gamma_k G) \otimes \Bbbk),$$

endowed with the inverse limit filtration, is called the *Malcev Lie algebra* of G (over $\Bbbk$).

- The group-algebra $\Bbbk G$ has a natural Hopf algebra structure, with comultiplication $\Delta(g) = g \otimes g$ and counit the augmentation map.

- (Quillen 1968) The I-adic completion of the group-algebra, $\widehat{\Bbbk G} = \varprojlim_k \Bbbk G / I^k$, is a filtered, complete Hopf algebra.

- An element $x \in \widehat{\Bbbk G}$ is called *primitive* if $\hat{\Delta} x = x \hat{\otimes} 1 + 1 \hat{\otimes} x$. The set of all such elements, with bracket $[x, y] = xy - yx$, and endowed with the induced filtration, is a complete, filtered Lie algebra.

- We then have

$$\mathfrak{m}(G) \cong \operatorname{Prim}(\widehat{\Bbbk G}).$$

$$\operatorname{gr}(\mathfrak{m}(G)) \cong \operatorname{gr}(G).$$

- (Sullivan 1977) The group G is 1-formal if and only if its Malcev Lie algebra is quadratic.

Graded and filtered formality

- The group G is *graded-formal* if its associated graded Lie algebra $\mathrm{gr}(G)$ is quadratic.
- The group G is *filtered formal* if its Malcev Lie algebra is filtered formal, i.e.,
$$\mathfrak{m}(G) \cong \widehat{\mathrm{gr}(\mathfrak{m}(G))}$$
- G is 1-formal $\Longleftrightarrow$ G is both graded-formal and filtered-formal.
- The group $G = \langle x_1, x_2 \mid [x_1, [x_1, x_2]] = 1 \rangle$ is filtered-formal. Yet G has a non-trivial 3MP of the form $\langle x_1, x_1, x_2 \rangle$. Hence, G is not graded-formal.
- The group $G = \langle x_1, \dots, x_5 \mid [x_1, x_2][x_3, [x_4, x_5]] = 1 \rangle$ is graded-formal. Yet G has a non-trivial 3MP of the form $\langle x_3, x_4, x_5 \rangle$. Hence, G is not filtered-formal.

Formality properties

Theorem (S.–Wang 2015)

Let $H \leqslant G$ be a subgroup which admits a split monomorphism $H \to G$. If G is graded-/filtered-/1-formal then H is graded-/filtered-/1-formal.

Theorem (SW)

Let G_1 and G_2 be two f.g. groups. TFAE:

- *G_1 and G_2 are graded-/filtered-/1-formal.*
- *$G_1 * G_2$ is graded-/filtered-/1-formal.*
- *$G_1 \times G_2$ is graded-/filtered-/1-formal.*

THEOREM (SW)

Suppose $\varphi\colon G_1 \to G_2$ *is a homomorphism between two f.g. groups, inducing an isomorphism* $H_1(G_1;\Bbbk) \to H_1(G_2;\Bbbk)$ *and an epimorphism* $H_2(G_1;\Bbbk) \to H_2(G_2;\Bbbk)$*. Then:*

- *If* G_2 *is* 1*-formal, then* G_1 *is also* 1*-formal.*
- *If* G_2 *is filtered-formal, then* G_1 *is also filtered-formal.*
- *If* G_2 *is graded-formal, then* G_1 *is also graded-formal.*

THEOREM (SW)

Let $\mathbb{K}/\Bbbk$ *be a field extension. A f.g. group* G *is graded-/filtered-/*1*-formal over* $\Bbbk$ *if and only if* G *is graded-/filtered-/*1*-formal over* $\mathbb{K}$*.*

EXPANSIONS IN GROUPS

- Let $\operatorname{gr}(\Bbbk G)$ be the associated graded algebra of $\Bbbk G$ with respect to the augmentation ideal, and let $\widehat{\operatorname{gr}}(\Bbbk G)$ be its degree completion.
- (D. Bar-Natan) A *multiplicative expansion* of a group G is a map

$$E\colon G \to \widehat{\operatorname{gr}}(\Bbbk G)$$

  such that the induced algebra morphism, $\bar{E}\colon \Bbbk G \to \widehat{\operatorname{gr}}(\Bbbk G)$, is filtration-preserving and induces the identity on associated graded algebras.
- Such a map E is called a *Taylor expansion* if it sends all elements of G to group-like elements of the Hopf algebra $\widehat{\operatorname{gr}}(\Bbbk G)$.

- G is said to be *residually torsion-free nilpotent* if any non-trivial element of G can be detected in a torsion-free nilpotent quotient.
- If G is finitely generated, the RTFN condition is equivalent to the injectivity of the canonical map $G \to \mathfrak{M}(G, \Bbbk)$.

Theorem (SW)

Let G be a finitely generated group. Then:

- *G is filtered-formal iff G has a Taylor expansion $G \to \widehat{\operatorname{gr}}(\Bbbk G)$.*
- *G is 1-formal iff G has a Taylor expansion and $\operatorname{gr}(\Bbbk G)$ is a quadratic algebra.*
- *G has an injective Taylor expansion iff G is residually torsion-free nilpotent and filtered-formal.*

Nilpotent groups and formality

- (Hasegawa 1989) A nilmanifold M^n is formal iff M is an n-torus.
- Let G be a finitely generated nilpotent group.
  - (Macinic–Papadima 2007) $\mathcal{V}^i(G) \subseteq \{1\}$.
  - (Macinic 2010) If G is q-formal, then $H^{\leqslant q+1}(G, \Bbbk)$ is generated by $H^1(G, \Bbbk)$. The converse holds if G is 2-step nilpotent.
- Let G be a finitely generated, torsion-free, nilpotent group.
  - (Carlson–Toledo 1995, Plantiko 1996) Suppose there is a non-zero decomposable element in the kernel of $\cup\colon H^1(G, \Bbbk) \wedge H^1(G, \Bbbk) \to H^2(G, \Bbbk)$; then G is not graded-formal.
  - (SW) Suppose G is filtered-formal. Then G is abelian if and only if $U(\operatorname{gr}(G, \Bbbk))$ is Koszul.
  - (SW) If G is 2-step nilpotent, and G_{ab} is torsion-free, then G is filtered-formal.

THEOREM (SW)

Let G be a finitely generated, filtered-formal group. Then all the nilpotent quotients $G/\gamma_i(G)$ are filtered-formal.

- Consequently, all the n-step, free nilpotent groups $F_k/\gamma_n F_k$ are filtered-formal.
- The unipotent groups $U_n(\mathbb{Z})$ of integer, upper triangular $n \times n$ matrices with 1's along the diagonal are filtered-formal, but not graded-formal for $n \geqslant 3$.
- All nilpotent Lie algebras of dimension 4 or less are filtered-formal (or, "Carnot").
- (Cornulier 2016) There is a 5-dimensional, 3-step nilpotent Lie algebra which is not filtered-formal.

SOLVABLE QUOTIENTS AND FORMALITY

THEOREM (SW)

Let G be a finitely generated group. For each $i \geqslant 2$, the quotient map $G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded $\Bbbk$-Lie algebras,

$$\operatorname{gr}(G, \Bbbk)/\operatorname{gr}(G, \Bbbk)^{(i)} \longrightarrow\!\!\!\!\!\rightarrow \operatorname{gr}(G/G^{(i)}, \Bbbk) .$$

Moreover,

- *If G is filtered-formal, then each solvable quotient $G/G^{(i)}$ is also filtered-formal, and the above map is an isomorphism.*
- *If G is 1-formal, then $\mathfrak{h}(G, \Bbbk)/\mathfrak{h}(G, \Bbbk)^{(i)} \cong \operatorname{gr}(G/G^{(i)}, \Bbbk)$.*

THEOREM (SW)

The quotient map $G \twoheadrightarrow G/G''$ induces a natural epimorphism of graded Lie algebras,

$$\operatorname{gr}(G,\Bbbk)/\operatorname{gr}(G,\Bbbk)'' \longrightarrow\!\!\!\!\!\rightarrow \operatorname{gr}(G/G'',\Bbbk) .$$

Moreover, if G is filtered-formal, this map is an isomorphism.

THEOREM (PAPADIMA–S. 2004, SW)

There is a natural epimorphism of graded Lie algebras,

$$\mathfrak{h}(G,\Bbbk)/\mathfrak{h}(G,\Bbbk)'' \longrightarrow\!\!\!\!\!\rightarrow \operatorname{gr}(G/G'',\Bbbk) .$$

Moreover, if G is 1-formal, then this map is an isomorphism.