Convex relaxation and variational approximation of functionals defined on 1-d connected sets

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The Steiner Tree Problem

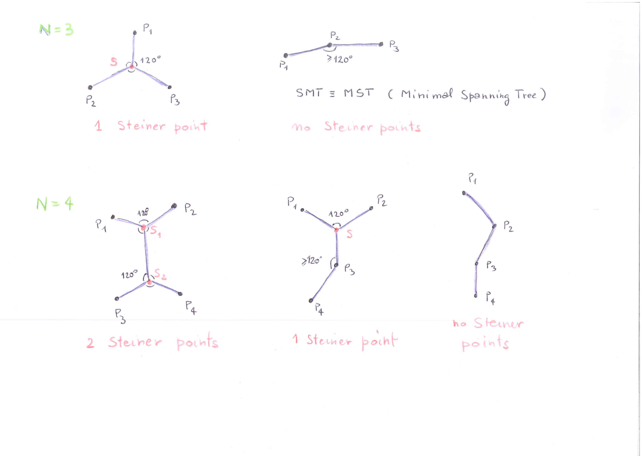
Steiner Tree Problem: Given N points $P_i \in X$ in a metric space, (e.g. X a graph, with P_i given vertices), find a connected (sub-)graph $F \subset X$ containing the points P_i and having minimal length.

An optimal graph F is called a Steiner Minimal Tree (SMT).

Examples. $X = \mathbb{R}^k$: Euclidean (or geometric) STP (design of optimal transport channels /networks w.r.t. given terminal points)
$X \subset \mathcal{G} \subset \mathbb{R}^k$ (contained) in a fixed grid $\mathcal{G}$ (or $X \subset \mathbb{R}^k$ endowed with the ℓ^1 metric): rectilinear STP (optimal design of net routing in VLSI circuits for $k = 2, 3$).

Euclidean STP is a NP-hard problem. Existence of PTAS, especially developed in case $k = 2$.

Euclidean STP - features of solutions



Acyclic graph, max $N - 2$ Steiner points (incident angles $\equiv 120°$)
No additional Steiner points $\Leftrightarrow$ SMT $\equiv$ MST (Minimal Spanning Tree, easy to compute)
Steiner ratio (MST/SMT) in $\mathbb{R}^2$: $2/\sqrt{3}$ (euclidean, open conj.), $2/3$ (rectilinear)

Formulation of the STP in a metric space X [Paolini-Stepanov]: given $A \subset X$ a compact (possibly infinite) set of terminal points,

$$(STP) \equiv \inf\{\mathcal{H}^1(S), S \textit{ connected}, S \supset A\}$$

Existence relies on Golab compactness theorem for compact connected sets. Allows for even further generalizations (e.g. $\inf \mathcal{H}^1(S)$, $S \cup A$ connected).

Functional framework not easy for computations.

STP vs Branched Optimal Transport.

Formulation for measures instead of sets: the network S connecting the P_i is made by streamlines of a vector measure (current) $\mu = \theta(x)\tau_S(x) \cdot \mathcal{H}^1 \llcorner S$ flowing unit masses located at P_i, $i < N$, to P_N. The transport cost is a sublinear (concave) function of the mass density, to favour branching [Xia]. For $0 < \alpha \leq 1$,

$$(M_\alpha) \equiv \inf \left\{ M_\alpha(\mu) = \int_S |\theta|^\alpha(x) d\mathcal{H}^1(x),\ \mathrm{div}\mu = (N-1)\delta_{P_N} - \sum_{i=1}^{N-1} \delta_{P_i} \right\}$$

Rmk. (M_1) is well-behaved, as a mass minimization problem, i.e. the minimization of the total variation norm $M_1(\mu) = ||\mu||$, it corresponds to an Optimal Transport Pb. with L^1 cost, (cf. Beckmann Pb.) while $(M_0) \equiv (STP)$ corresponds to size minimization (minimizing sequences a priori non compact).
Existence for $\alpha > 0$: [Xia], [Bernot-Caselles-Morel], [Depauw-Hardt].

Expected convergence $(M_\alpha) \to (M_0)$ as $\alpha \to 0$
(cf. [Marchese-Massacesi])

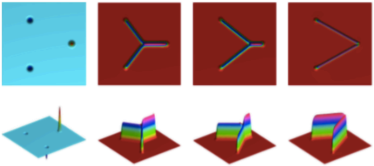


Figure 1: Optimal irrigation of two discrete measures by one single source for $\alpha = 0.1, 0.6$ and 0.95

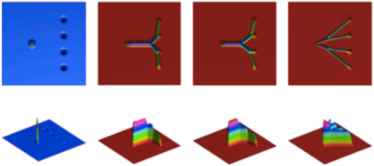


Figure 2: Optimal irrigation of four discrete measures by one single source for $\alpha = 0.1, 0.6$ and 0.95

(picture from [Oudet-Santambrogio])

Approximation of (M_0) in $\mathbb{R}^2$ by $F_\epsilon(\mu) = M_0(\mu) + \epsilon^2 M_1(\mu)$ [Depauw-Hardt], [Morgan]

Variational approximation (via Γ-convergence) of (M_α) through phase transition functionals defined for $u \in H^1(\mathbb{R}^2;\mathbb{R}^2)$ [Oudet-Santambrogio]

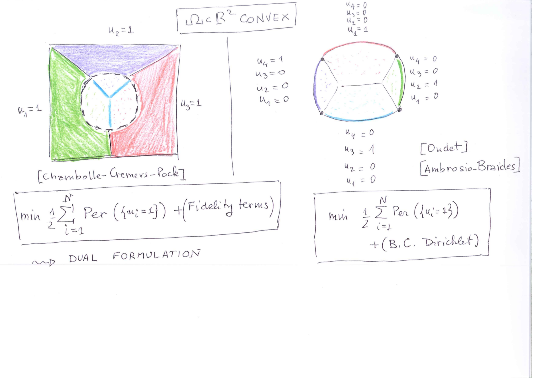
$$M_{\alpha,\epsilon}(u) = \epsilon^{\alpha-1}\int_{\mathbb{R}^2}|u|^\beta + \epsilon^{\alpha+1}\int_{\mathbb{R}^2}|\nabla u|^2 \quad (\operatorname{div} u = \rho_0 - \rho_1)$$

Approximation of minimizers of (M_0) by minimizers of phase transition functionals in $\mathbb{R}^2$ [Bonnivard-Lemenant-Santambrogio], [Millot & al.], [Chambolle-Merlet-Ferrari]

$$F_\epsilon(\rho) = \frac{1}{4\epsilon}\int_{\mathbb{R}^2}(1-\rho)^2 + \epsilon\int_{\mathbb{R}^2}|\nabla\rho|^2 + \frac{1}{c_\epsilon}\sum_{i=1}^{N} d_\rho(x_i, x_N)$$

where $d_\rho(x_i, x_N) = \inf\{\int_\gamma \rho(x)d\mathcal{H}^1(x),\ \gamma(0) = x_i,\ \gamma(1) = x_N\}$. Level sets $\{d_\rho = 0\}$ are connected and $d_{\rho_\epsilon} \to d$ with $\{d = 0\} \equiv$ SMT.

If $P_i \in \partial\Omega$, $\Omega \subset \mathbb{R}^2$ convex, (STP) is related to a minimal partition problem, e.g.
$\inf\{\int_\Omega |\nabla u|,\ u \in BV(\Omega; \{e_1, ..., e_N\}),\ u_{|\partial\Omega} = u_0\}$ [Ambrosio-Braides]



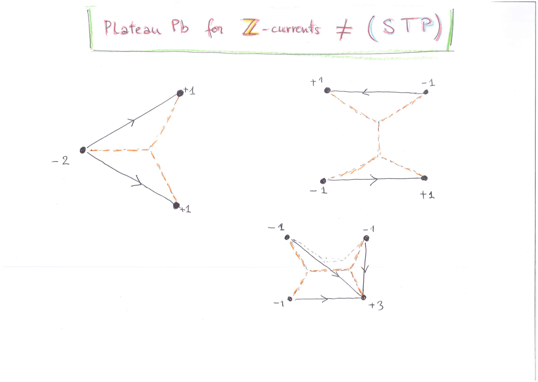
Variants, approximations, convex relaxation and dual formulation:
[Otto et al.], [Oudet], [Bretin et al.], [Chambolle-Cremers-Pock]

Interpretation of area minimizing surfaces as solutions of a Plateau problem for currents in a suitable covering space of $\mathbb{R}^k$.
Use of the calibration method [Brakke],

The case $k = 2$ corresponds to STP: analysis and variational approximation [Bellettini-Amato-Paolini]

Caibrations [Pluda-Carioni]

Let's try to formulate (STP) as a Plateau problem for polyhedral (or rectifiable) 1-chains $T = \sum L_i$ with $\partial T = \sum a_i P_i$
Integer multiplicities $a_i \in \mathbb{Z}$ are not suited: Plateau problem corresponds to an OT problem $(M_1) \equiv \inf ||T||$, with $||T|| = \sum |L_i|$.



Some examples of troubles: non connectedness, no Steiner (branching) points...

Plateau problem for G-currents vs STP

The approach of [Marchese-Massaccesi]

Let's try with a more general discrete coefficient group G: what should be the requirements on G?

G normed abelian group (e.g. $G < E$ additive subgroup of a Banach space E)

$$T = \sum \gamma_j L_j, \quad \partial T = \sum \gamma_j \partial L_j = \sum_{i=1}^{N} g_i P_i, \quad g_i \in G, \quad \gamma_j = \sum_{i \in \Lambda_j} g_i$$

$$||T|| = \sum ||\gamma_j|| \cdot |L_j|, \qquad ||\gamma_j|| = 1 \; \forall j \; \Rightarrow \; ||T|| = \sum |L_i|$$

$$\sum_{i=1}^{N} g_i P_i = \partial T \Longleftrightarrow \sum_{i=1}^{N} g_i = 0 \qquad \text{(boundary)}$$

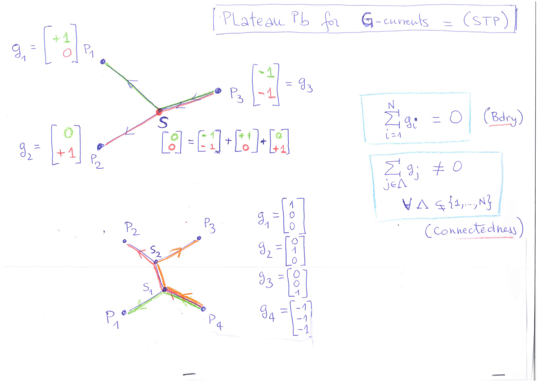
$$\sum_{i \in \Lambda} g_i \neq 0 \quad \forall \Lambda \subset \{1, ..., N\}, \; \Lambda \neq \{1, ..., N\} \qquad \text{(connectedness)}$$

$|| \sum_{i \in \Lambda} g_i || = 1$ ensures both connectedness and $||T|| = \sum |L_i|$.

Plateau problem for G-currents vs STP

The approach of [Marchese-Massaccesi]

Consider for example $E = \mathbb{R}^{N-1}$, $G = \mathbb{Z}^{N-1}$, $g_i = e_i$ for $i = 1, ..., N-1$, $g_N = -\sum_{i=1}^{N-1} e_i$



Remark. Endowing E^+ (positive orthant of E) with the ℓ^∞ norm fulfills all previous requirements!

Remark. Endowing E^+ with the ℓ^q norm fulfills all requirements for an equivalent formulation of the irrigation problem (M_α), with $\alpha = q^{-1}$.

Remark. any norm on E that coincides with ℓ^∞ (resp. ℓ^q) on E^+ is suited to handle $(M_0) \equiv STP$ (resp. (M_α)).

A natural choice, having in mind optimal convex relaxations of the problem, is to considet the largest possible extension to E (convex 1-homogeneous envelope) of $\ell^\infty_{|E^+}$ (resp. $\ell^q_{|E^+}$)

This envelope coincides with the norm introduced by [Marchese-Massaccesi] to study (M_0) (resp. (M_α)) via the calibration method.

Rectifiable G-currents

Original definition by [Fleming], generalization to metric spaces by [Depauw-Hardt]. We follow [Marchese-Massaccesi].

Let $G < E$ be a discrete subgroup of a (m-dimensional) Banach space E, $R \subset \mathbb{R}^k$ a (closed) d-rectifiable set, $\tau(x) \in \Lambda_*(\mathbb{R}^k)$ a $\mathcal{H}^d$-measurable orientation for R (a unit simple d-vectorfield tangent to R), and $g(x) : R \to G \subset E$ a $\mathcal{H}^d$-measurable G-valued multiplicity function defined on R. The vector measure T (with spt $T = R$)

$$T \equiv T(g, \tau, R) \equiv g(x) \otimes \tau(x) \cdot \mathcal{H}^d \llcorner R$$

is a rectifiable G-current. It is a limit in $(C_c^1)^*$ of polyhedral G-chains.

Rmk. If e_j, $j = 1, ..., m$ is a basis for E, with $||e_j|| = 1$, then we may write $g(x) = \sum_j g^j(x) e_j$ with $g^j(x) \in \mathbb{Z}$ and accordingly $T = \sum_j T^j e_j$, with $T^j = g^j(x)\tau(x) \cdot \mathcal{H}^d \llcorner R$ a (cassical) rectifiable current.

Normal (resp. integral) E- (resp. G-) currents

E-currents T are defined by duality with smooth compactly supported E^*-valued forms $\omega(x) = \theta(x) \otimes \phi(x) = \sum_j \omega_j(x) e^j \in E^* \otimes \Lambda^*(\mathbb{R}^k)$.

Exterior derivative $d\omega(x) = \sum_j d\omega_j(x) e^j$

Mass norm $||T|| = \sup\{T(\omega),\ \ ||\theta||_{E^*} \leq 1,\ \ ||\phi||_* \leq 1\}$

Boundary $\partial T(\omega) = T(d\omega)$. **Rmk.** $\partial T = \sum_j \partial T^j e_j$.

$||T|| < +\infty \Rightarrow T = (\sum_i g_i \otimes \tau_i)|\mu_T|$,
$T(\omega) = \int \sum_i \langle \theta(x), g_i(x) \rangle \cdot \langle \phi(x), \tau_i(x) \rangle d|\mu_T|$

Normal currents: $N(T) = ||T|| + ||\partial T|| < +\infty$

Integral currents: both T and ∂T rectifiable G-currents

$T(\omega) = \int_{sptT} \langle \theta(x), g(x) \rangle \cdot \langle \phi(x), \tau(x) \rangle d\mathcal{H}^d(x)$

$||T|| = \int_{sptT} ||g(x)|| d\mathcal{H}^d(x)$

$(C^1_c)^*$ closure and compactness theorem for N-bdd normal and integral currents: apply componentwise [Federer-Fleming]

Plateau Pb for Normal and Integral currents vs (M_α)

Given S a d-rectifiable G-current, **existence** of mass-minimizers for the Plateau problems

$$(M_G) \equiv \inf\{||T||, \quad T \text{ integral } G\text{-current}, \ \partial T = S\}$$

$$(M_E) \equiv \inf\{||T||, \quad T \text{ normal } E\text{-current}, \ \partial T = S\}$$

(M_E) is a convex relaxation of (M_G)

Rmk. In case $S = \sum_{i=1}^N g_i \otimes \delta_{P_i}$, (M_G) is equivalent to (M_α) in view of

Lemma (M-M)

For any compact connected set $K \subset \mathbb{R}^k$ s.t. $K \supset \{P_1, ..., P_N\}$ and $\mathcal{H}^1(K) < +\infty$, $\exists T$ 1-rectifiable G-current s.t. $\partial T = S$, spt T connected and spt $T \subset K$.

Rmk. Structure of 1-currents: (nonunique) decomposition in acyclic and cyclic part. $T = \sum_j U_j + \sum_\ell C_\ell$, with $||T|| = \sum_j ||U_j|| + \sum_\ell ||C_\ell||$ and $\partial C_\ell = 0$. In particular, solutions of (M_α) are acyclic.

Plateau Pb for Normal and Integral currents vs (M_α)

Rmk. For $G = \mathbb{Z}$ and $E = \mathbb{R}$, (M_G) and (M_E) are equivalent in case T is a 1-current [Smirnov], [Paolini-Stepanov] or a $(k-1)$-current (coarea formula). Much less is known for d-currents, $1 < d < k-1$, and also for $1 \leq d \leq k-1$ for general G. In particular, the conjecture $(M_G) \equiv (M_E)$ is open for 1-currents in $\mathbb{R}^k$.

Rmk. $(M_G) \equiv (M_E)$ if the normal E-current T decomposes as $T = \int_\Lambda T_\lambda d\lambda$ and $\partial T = \int_\Lambda \partial T_\lambda d\lambda$ with T_λ integral G-currents, with $||T|| = \int_\Lambda ||T_\lambda|| d\lambda$ (true in the classical case $E = \mathbb{R}$ and $G = \mathbb{Z}$: Smirnov decomposition of solenoidal vector fields into elementary solenoids)

Rmk. $(M_G) \equiv (M_E)$ if and only if the following homogeneity property holds for integral G-currents T with fixed boundary:

$$\inf\{||T||,\ \partial T = nS\} = n \cdot \inf\{||T||,\ \partial T = S\} \qquad \forall n \in \mathbb{N}\,.$$

$(M_G) \equiv (M_E)$ follows in combination with the approximation of E-currents by rational multiples of polyhedral G-currents.

Calibrations

A method to prove that $(M_G) = (M_E)$ for some given boundary S is to construct a **calibration** for the minimizers of (M_G).

Let T_0 be a minimizing G-current. An element φ of the dual space of T_0 (a generalized E^*-valued differential form) is a calibration if $\langle \varphi, T_0 \rangle = ||T_0||$, $||\varphi||_* \leq 1$, $\langle \varphi, \partial R \rangle = 0$ for every E-boundary ∂R.
In fact, for any E-current T s.t. $\partial T = S = \partial T_0$ there exists R s.t. $T = T_0 + \partial R$. Hence

$$||T_0|| = \langle \varphi, T_0 \rangle = \langle \varphi, T_0 + \partial R \rangle = \langle \varphi, T \rangle \leq ||\varphi||_* ||T|| \leq ||T||$$

and T_0 is also minimizing among normal currents. The method is used when the candidate minimizing G-current T_0 is known.

Remark. In [Marchese-Massaccesi] some examples of (generalized) calibrations are constructed for STP with terminal points at the vertices of an equilateral triangle, of a square, of a hexagon and at the vertices of a hexagon together with its center.
In [Massaccesi-Oudet-Velichkov] an approximation scheme is implemented to find them numerically, and tested in the above cases.

Convex relaxation of (M_α) in $\mathbb{R}^k$ [BOO]

Let S be an integral G-boundary. Interested in $S = \sum_{i=1}^{N} g_i \delta_{P_i}$,

Let T_0 be an integral G-current s.t. $\partial T_0 = S$. For any other T normal E-current s.t. $\partial T = S$, we have $\partial T = \partial T_0$, hence in $\mathbb{R}^k$ there exists an E-current R such that $T = T_0 + \partial R$.

Rmk. If T is an integral G-current then we may choose R to be integral (by Federer deformation Thm).

Hence we may reformulate (M_G) and (M_E) as follows

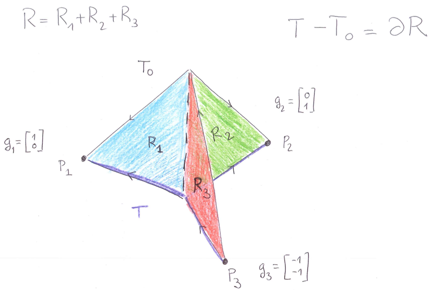
$$(M_G) \equiv \inf\{\mathcal{F}(R) = ||T_0 + \partial R||, \quad R \text{ integral } G\text{-current}\}$$
$$(M_E) \equiv \inf\{\mathcal{F}(R) = ||T_0 + \partial R||, \quad R \text{ normal } E\text{-current}\}$$

Remark. For $k = 2$, $\partial R = \star du$, with $u \in BV(\mathbb{R}^2; \mathbb{Z})$.

Remark. ([Alberti-Baldo-O]) ∂R integral 1-(classical) boundary in $\mathbb{R}^k$, $k \geq 3 \Leftrightarrow \exists u \in W^{1,p}(\mathbb{R}^k; \mathbb{R}^{k-1})$, $|u| = 1$ a.e. , s.t. $\partial R = Ju$, where $Ju = \frac{1}{k-1} \star d(u^*(\rho^{k-1} vol_{S^{k-2}})) = \frac{1}{k-1} \star d(\sum_{j=1}^{k-1} (-1)^{j+1} u_j \widehat{du_j})$.

Convex relaxation of (M_α) in $\mathbb{R}^k$

Example. $S = e_1(\delta_{P_1} - \delta_{P_3}) + e_2(\delta_{P_2} - \delta_{P_3})$ in $\mathbb{R}^k$, $k = 2, 3$.



Case k=3. $\partial R = Ju_1 e_1 + Ju_2 e_2, \quad u_i \in W^{1,p}(\mathbb{R}^3; S^1)$,

$$Ju_i = \frac{1}{2}\nabla \times (u_i^1 \nabla u_i^2 - u_i^2 \nabla u_i^1)$$

Case k=2. $\partial R = \nabla^\top u_1 e_1 + \nabla^\top u_2 e_2, \quad u_i \in BV(\mathbb{R}^2; \mathbb{Z})$

Revisiting (M_G)**.** Let $T_0 = (\sum_{j=1}^{N-1} g^j e_j) \otimes \tau\mathcal{H}^1 \llcorner sptT = \sum_{j=1}^{N-1} \mu_j e_j$, and $R = \sum_{j=1}^{N-1} Ju_j e_j$. Denote $U = (u_1, ..., u_{N-1})$, we have, for $k \geq 3$,

$$\mathcal{F}(R) = \mathcal{F}(U) = \int \sup_{1 \leq j \leq N-1} |\mu_j + Ju_j|$$

where $|\mu_j + Ju_j|$ is the total variation of the vector measure $\mu_j + Ju_j$.

In the case $\mathbf{k} = \mathbf{2}$ we obtain

$$\mathcal{F}(R) = \mathcal{F}(U) = \int \sup_{1 \leq j \leq N-1} |\mu_j + \nabla^\top u_j|$$

with $u_j \in BV(\mathbb{R}^2; \mathbb{Z})$, i.e. a minimal partition problem with drift term μ_j.

Remark. If $P_i \in \partial\Omega$, $\Omega \subset \mathbb{R}^2$ convex, take T_0 s.t. spt $T_0 \cap \Omega = \emptyset$. Minimizing $\mathcal{F}(U)$ reduces to the minimal partition problem

$$\mathcal{F}_\Omega(U) = \int_\Omega \sup_{1 \leq j \leq N-1} |\nabla u_j|$$

for $u_j \in BV(\Omega; \mathbb{Z})$, with suitable Dirichlet boundary conditions

Convex relaxation of (M_α) in $\mathbb{R}^k$

Revisiting (M_E)**.** Write the normal current $R = \sum_{j=1}^{N-1} R^j e_j$, set $R^j = \star\omega_j$ with ω_j a measure-valued $(k-2)$-differential form. We have $\partial R_j = \star d\omega_j$. In the same way let $\mu_j = \star\eta_j$ and let $\xi_j = *\eta_j$ (1-form) and $\psi_j = *\omega_j$ (2-form).

Denote $\Omega = \sum_j \omega_j e_j$, $\Psi = \sum_j \psi_j e_j$, recall $d^* = - * d *$ on 2-forms. We have

$$\mathcal{F}(\Omega) = \mathcal{F}(\Psi) = \int \sup_{1\le j\le N-1} |\star(\eta_j + d\omega_j)| = \int \sup_{1\le j\le N-1} |\xi_j - d^*\psi_j|$$

for $\Omega, \Psi \in L^1$ with $d\Omega, d^*\Psi \in \mathcal{M}$.

Recall: for a 2-form $\theta = \sum_{1\le i<j\le k} \theta^{ij} dx_i \wedge dx_j$ in $\mathbb{R}^k$ we have

$$d^*\theta = \sum_{i=1}^{k} \left(\sum_{1\le j<i} \frac{\partial\theta^{ji}}{\partial x_j} - \sum_{i<j\le k} \frac{\partial\theta^{ij}}{\partial x_i} \right) dx_i \,.$$

Convex relaxation of (M_α) in $\mathbb{R}^k$

Examples in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Case k $= 2$. We have $\mathcal{F}(\Omega) = \int \sup_{1 \le j \le N-1} |\mu_j + \nabla^\top \omega_j)|$

for $\omega_j \in BV(\mathbb{R}^2)$. $\quad (M_E) = \inf\{\mathcal{F}(\Omega),\ \Omega \in BV\}$

Case k $= 3$. We have $\mathcal{F}(\Omega) = \int \sup_{1 \le j \le N-1} |\mu_j + \nabla \times \omega_j)|$

for $\omega_j \in L^1(\mathbb{R}^3; \mathbb{R}^3)$, and $\nabla \times \omega_j \in \mathcal{M}(\mathbb{R}^3; \mathbb{R}^3)$.

$$(M_E) = \inf\{\mathcal{F}(\Omega),\ \Omega \in L^1,\ d\Omega \in \mathcal{M}\}$$

Numerical approach. We develop a discrete approximation scheme for $(M_E) = \inf \mathcal{F}(\Omega) = \inf \mathcal{F}(\Psi)$ based on proximal operators and alternate projections. Remark that this gives also a discrete approximation scheme for the mean curvature flow of networks with given terminal points in the non parametric framework.

Dual formulation

Also the dual formulation is suitable to exploit efficient discrete approximation schemes as e.g. in [Chambolle-Cremers-Pock].

Let $\Xi = \sum_j \xi_j e_j$ the matrix-valued measure form representing T_0, and $d^*\Psi$ the one representing ∂R. Let $\Phi = \sum_j \phi_j e_j$ be a test form. Then

$$\begin{aligned}\mathcal{F}(\Psi) &= \sup_{1\le j\le N-1} |\xi_j - d^*\psi_j|(\mathbb{R}^k) \\ &= \|\Xi - d^*\Psi\| = \sup\{\langle \Xi - d^*\Psi, \Phi\rangle\,,\ ||\Phi||_* \le 1\} \\ &= \sup\{\langle \Xi, \Phi\rangle + \langle \Psi, d\Phi\rangle\,,\ ||\Phi||_* \le 1\}\end{aligned}$$

where $||\Phi||_* = \sup_x |\sum_{j=1}^{N-1} \phi_j(x)|$.

Remark. This formulation extends (and rigorously justifies!) the dual formulation for the minimal partition problem in a convex subset of $\mathbb{R}^2$ proposed by [Chambolle-Cremers-Pock].

Reformulation of (M_E) as a Beckmann-type Problem

Recall that given two distributions f_0 and f_1 of equal mass in $\mathbb{R}^k$, the Beckmann optimal allocation problem consists in finding

$$\inf \left\{ \int_{\mathbb{R}^k} |\mu|, \;\; \mathrm{div}\, \mu = f_1 - f_0 \right\}$$

among μ vector Radon measures on $\mathbb{R}^k$.

For $S = \sum_{j=1}^{N-1} e_j \otimes (\delta_{P_j} - \delta_{P_N})$, let $T = \sum_j \mu_j e_j$ a normal E-current (*emph.* μ_j vector measures). Condition $\partial T = S$ translates into

$$\mathrm{div}\, \mu_j = \delta_{P_j} - \delta_{P_N}, \qquad j = 1, ..., N-1$$

and (M_E) translates into

$$\inf \left\{ \int \sup_j |\mu_j|, \quad \mathrm{div}\, \mu_j = \delta_{P_j} - \delta_{P_N} \quad \forall\, 1 \leq j \leq N-1 \right\},$$

to be compared with the Beckmann - OT problem

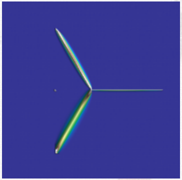
$$\inf \left\{ \int \sum_j |\mu_j|, \qquad \mathrm{div} \sum_j \mu_j = (\sum_j \delta_{P_j}) - (N-1)\delta_{P_N} \right\}$$

A discrete convex setting based on Chambolle-Cremers-Pock

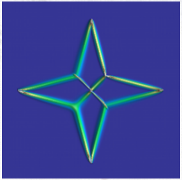
Non uniquement provides convexe combination of minimizers

$$\min_{(u_i^h)_{1\le i<N}\in L,\,(\psi_P^h)_{P\subset\{1,\dots,N-1\}}\in(\mathbb{R}^{2\mathcal{T}})^{2^{N-1}}} \frac{h^2}{2}\sum_{t\in\mathcal{T}}\sum_{P\subset\{1,\dots,N-1\}} |(\psi_P^h)_t|$$

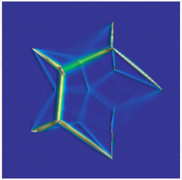
$$(\nabla^h u_i^h)_t = \sum_{P\subset\{1,\dots,N-1\},\,i\in P} (\psi_P^h)_t$$



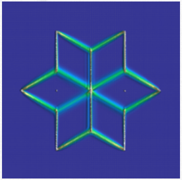
N=3



N=4



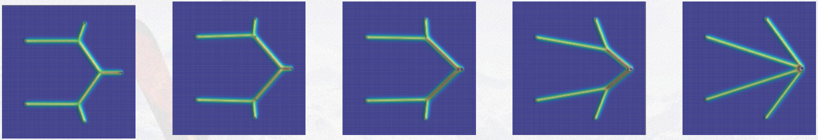
N=5



N=7

Link with irrigation problem

$$\int_{\Omega\setminus\gamma_i} f_h^i(u_i^h) = \int_{\Omega\setminus\gamma_i} |Du_i^h|^2 + \frac{1}{h^2} W(u_i^h)$$



$$\mathcal{G}_h^\alpha(u_i^h) = \int_{\Omega\setminus\gamma_i} h \left(\sum_{i=1}^{N-1} f_h^i(u_i^h)^{1/\alpha} \right)^\alpha$$

Variational approximation for $(M_G) \equiv (M_\alpha)$

Goal: find functionals $\mathcal{F}_\epsilon(U)$ of Ginzburg-Landau type that approximate $\mathcal{F}(U)$ in the sense of Γ-convergence. Recall some facts:

Case k $\geq$ 3: define $E_\epsilon(u) = \int_\Omega |\nabla u|^{k-1} + \frac{1}{\epsilon^2}(|u|^2-1)^2$, with $u \in W^{1,k-1}(\Omega \subset \mathbb{R}^k; \mathbb{R}^{k-1})$. Then for any sequence (u_ϵ) of minimizers of E_ϵ (under given constraints) there exists $u \in W^{1,p}(\Omega; S^{k-2})$ s.t. (up to a subsequence)

$$Ju_\epsilon \to Ju \quad \text{in } (C_0^1(\Omega))^*, \quad \frac{1}{|\log \varepsilon|} E_\epsilon(u_\epsilon) \to |Ju|(\Omega)$$

and Ju is a mass-minimizing integral boundary in Ω (under corresponding limiting constraints) [Alberti-Baldo-O]. Cf. also [Sandier] in case $\Omega = \mathbb{R}^k \setminus \{P_1, N_1, ..., P_\ell, N_\ell\}$.

Case k $=$ 2: define $E_\epsilon(u) = \int_\Omega |\nabla u|^2 + \frac{1}{\epsilon^2} \sin^2(\pi u)$, with $u \in W^{1,2}(\Omega \subset \mathbb{R}^2)$. Then, up to a subsequence of (constrained) minimizers (u_ϵ) of E_ϵ there exists $u \in BV(\Omega; \mathbb{Z})$ s.t.

$$u_\epsilon \to u \quad \text{in } L^1, \quad \epsilon E_\epsilon(u_\epsilon) \to c_0 |\nabla u|(\Omega)$$

and ∇u is a (constrained) minimizer of the TV in Ω [Modica-Mortola].

Variational approximation for $(M_G) \equiv (M_\alpha)$ [BOO]

Case k $= 2$: For $\mu = \tau \mathcal{H}^1 \llcorner \gamma$ a multiplicity one rectifiable 1-current in $\mathbb{R}^2$, define $E^\mu_\epsilon(u) = \int_{\mathbb{R}^2} e^\mu_\epsilon(u) dx = \int_{\mathbb{R}^2} |\mu + \nabla^\top u|^2 + \frac{1}{\epsilon^2} \sin^2(\pi u)$, with $u \in W^{1,2}(\Omega \subset \mathbb{R}^2)$. Then, up to a subsequence of minimizers (u_ϵ) of E^μ_ϵ there exists $u \in BV(\mathbb{R}^2; \mathbb{Z})$ s.t.

$$u_\epsilon \to u \quad \text{in } L^1, \qquad \epsilon E^\mu_\epsilon(u_\epsilon) \to c_0 |\mu + \nabla^\top u|(\mathbb{R}^2)$$

and u minimizes $|\mu + \nabla^\top u|(\mathbb{R}^2)$ on $BV(\mathbb{R}^2; \mathbb{Z})$ (cf. [Baldo-O])

Variational approximation for $(M_G) \equiv (M_\alpha)$

Proposition ([BOO] Γ-convergence, $k = 2$)

Let $P_1, ..., P_N \in \mathbb{R}^2$, $S = \sum_{j=1}^{N-1} e_j \otimes (\delta_{P_j} - \delta_{P_N})$, $T_0 = \sum_{j=1}^{N-1} \mu_j e_j$ s.t. $\partial T_0 = S$.
For $U = (u_1, .., u_{N-1}) \in W^{1,2}(\mathbb{R}^2; \mathbb{R}^{N-1})$ let

$$\mathcal{F}_\epsilon(U) = \int_{\mathbb{R}^2} \sup_{1 \le j \le N-1} e_\epsilon^{\mu_j}(u_j)\, dx.$$

Let U_ϵ be a minimizer of $\mathcal{F}_\epsilon$. Then, up to a subsequence,

$U_\epsilon \to U$ in $L^1(\mathbb{R}^2; \mathbb{R}^{N-1})$, $\epsilon \mathcal{F}_\epsilon(U_\epsilon) \to c_0 \mathcal{F}(U)$ and

U minimizes $\mathcal{F}$ on $BV(\mathbb{R}^2; \mathbb{Z}^{N-1})$.

Variational approximation for $(M_G) \equiv (M_\alpha)$ [BOO]

Case k $= 3$: Let $T_0 \equiv \mu = \tau \mathcal{H}^1 \llcorner \gamma$ be an integral current without boundary in $\Omega \subset \mathbb{R}^3$, so that in particular we have, for a good cover $\{V_\ell\}$ of Ω, $\mu \llcorner V_\ell = Jv^\ell = \star dA^\ell$, with $A^\ell = v^\ell \times dv^\ell$, $v^\ell \in W^{1,p}(V_\ell; \mathbb{C})$, $|v^\ell| = 1$. The expression $\nabla_A u = \nabla u + iA^\ell u$ on V_ℓ defines globally on $\Omega \setminus \{P_1, ..., P_N\}$ a covariant derivative $\nabla_A u$. Let

$$E_\epsilon^\mu(u) = \int_\Omega e_\epsilon^\mu(u)dx = \int_\Omega |\nabla_A u|^2 + \frac{1}{\epsilon^2}(1 - |u|^2)^2$$

for $u \in W^{1,2}(\Omega; \mathbb{C})$. Let (u_ϵ) be a sequence of minimizers of E_ϵ^μ. There exists $u \in W^{1,p}(\Omega; S^1)$ s.t. (up to a subsequence)

$$Ju_\epsilon \to Ju \quad \text{in } (C_0^1(\Omega))^*, \quad \frac{1}{|\log \varepsilon|} E_\epsilon(u_\epsilon) \to |\mu + Ju|(\Omega)$$

and $T = Ju + \mu = Ju + T_0$ is a mass-minimizing integral current without boundary in Ω in the integral homology class of T_0.

(cf. [Alberti-Baldo-O, Baldo-O, BOO])

Variational approximation for $(M_G) \equiv (M_\alpha)$

Proposition ([BOO] Γ-convergence, $k = 3$)

Let $P_1, ..., P_N \in \mathbb{R}^3$, $S = \sum_{j=1}^{N-1} e_j \otimes (\delta_{P_j} - \delta_{P_N})$, $T_0 = \sum_{j=1}^{N-1} \mu_j e_j$ *s.t.* $\partial T_0 = S$.
For $U = (u_1, .., u_{N-1}) \in W^{1,2}(\mathbb{R}^3; \mathbb{C}^{N-1})$ *let*

$$\mathcal{F}_\epsilon(U) = \int_{\mathbb{R}^3} \sup_{1 \le j \le N-1} e_\epsilon^{\mu_j}(u_j)\, dx.$$

Let U_ϵ *be a minimizer of* $\mathcal{F}_\epsilon$. *Then, up to a subsequence,*

$JU_\epsilon \to JU$ *in* $(C_0^1(\mathbb{R}^3))^*$, $\frac{1}{|\log \varepsilon|} \mathcal{F}_\epsilon(U_\epsilon) \to \mathcal{F}(U)$ *and*

U *minimizes* $\mathcal{F}$ *on* $W^{1,p}(\mathbb{R}^3; (S^1)^{N-1})$.

Case k $\geq$ 3: Let $-T_0 \equiv \mu = \tau \mathcal{H}^1 \llcorner \gamma$ be an integral current without boundary in $\Omega \subset \mathbb{R}^k$, so that in an open neighborhood $W \subset \Omega$ of spt T_0, $\mu \llcorner W = Jv$, with $v \in W^{1,p}(V; \mathbb{R}^{k-1})$, $|v| = 1$.
Let $B = \mathbf{1}_W \nabla v$ and define

$$E_\epsilon^\mu(u) = \int_\Omega e_\epsilon^\mu(u)dx = \int_\Omega |\nabla u - B|^{k-1} + \frac{1}{\epsilon^2}(1 - |u|^2)^2$$

for $u \in W^{1,k}(\Omega; \mathbb{R}^{k-1})$. Let (u_ϵ) be a sequence of minimizers of E_ϵ^μ.
There exists $u \in W^{1,p}(\Omega; S^{k-2})$ s.t. (up to a subsequence)

$$Ju_\epsilon \to Ju \quad \text{in } (C_0^1(\Omega))^*, \quad \frac{1}{|\log \varepsilon|} E_\epsilon(u_\epsilon) \to |Ju - \mu|(\Omega)$$

and $T = Ju - \mu = Ju + T_0$ is a mass-minimizing integral current without boundary in Ω in the integral homology class of T_0.

(cf. [ABO, BOO])

Variational approximation for $(M_G) \equiv (M_\alpha)$

Proposition ([BOO] Γ-convergence, $k \geq 3$)

Let $P_1, ..., P_N \in \mathbb{R}^k$, $S = \sum_{j=1}^{N-1} e_j \otimes (\delta_{P_j} - \delta_{P_N})$, $T_0 = \sum_{j=1}^{N-1} \mu_j e_j$ s.t. $\partial T_0 = S$.
For $U = (u_1, .., u_{N-1}) \in W^{1,k-1}(\mathbb{R}^k; \mathbb{R}^{k(N-1)})$ let

$$\mathcal{F}_\epsilon(U) = \int_{\mathbb{R}^k} \sup_{1 \leq j \leq N-1} e_\epsilon^{\mu_j}(u_j)\, dx.$$

Let U_ϵ be a minimizer of $\mathcal{F}_\epsilon$. Then, up to a subsequence,

$JU_\epsilon \to JU$ in $(C_0^1(\mathbb{R}^k))^*$, $\frac{1}{|\log \varepsilon|}\mathcal{F}_\epsilon(U_\epsilon) \to \mathcal{F}(U)$ and

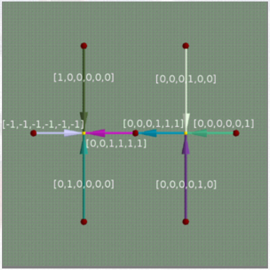
U minimizes $\mathcal{F}$ on $W^{1,p}(\mathbb{R}^k; (S^{k-1})^{N-1})$.

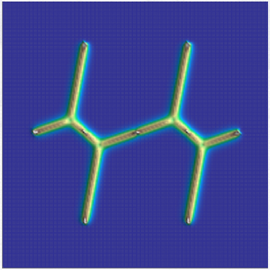
Gamma-convergence and first results associated to this formulation (non convex setting, local minima)

$$\int_{\Omega\setminus\gamma_i} f_h^i(u_i^h) = \int_{\Omega\setminus\gamma_i} |Du_i^h|^2 + \frac{1}{h^2} W(u_i^h)$$

$$\mathcal{G}_h^0(u_i^h) = \int_{\Omega\setminus\gamma_i} h \sup_{1\le i\le N-1} f_h^i(u_i^h)$$

Standard examples





Harmonic maps with prescribed degree and (M_α)

[Baldo-O] '17

Recall the relaxation result for harmonic sphere-valued maps with prescribed degree ([Brezis-Coron-Lieb], [Almgren-Browder-Lieb])

$$\inf_V \left\{ \int_{\mathbb{R}^k} |Du|^{k-1} \right\} = c_k |P - Q|$$

$P, Q \in \mathbb{R}^k$, $V = \{u \in W^{1,k}(S^{k-1}), \deg(u, P) = +1, \deg(u, Q) = -1\}$

Proposition ([Baldo-O] '17)

Let $P_1, ..., P_N \in \mathbb{R}^k$ and define, for $i = 1, ..., N-1$,
$V_i = \{u \in W^{1,k}(\mathbb{R}^k; S^{k-1}), \deg(u, P_i) = +1, \deg(u, P_N) = -1\}$,

$$\mathcal{E}(U) = \int_{\mathbb{R}^k} \sup_{1 \le i \le N-1} |Du_i|^{k-1}, \quad U = (u_1, ..., u_{N-1}) \in V \equiv \Pi_i V_i$$

We then have $\inf_V \mathcal{E}(U) = c_k(M_0)$

Rmk. Generalizations to (M_α) and to arbitrary dimensions/ambients

Some concluding remarks

Remark. Previous theory may be generalized to solve variants of STP, and (M_α), e.g. rectilinear STP, (M_α) on manifolds (cyclic part has to be taken into account), on sufficiently nice metric spaces (e.g. manifolds with densities),...

Remark. Useful formulation of the size minimization problem in homology classes of manifolds (cf. [Morgan])

Remark. Numerical implementation for $\mathcal{F}_\epsilon$ in case $k = 2, 3$ cf. [Bretin et al.], [Oudet] for $k = 2$

Remark. Find analogies (if any) with dynamical models of transport (e.g. colliding/sticky particles with mass absorbtion)

Remark. General open question: prove, disprove, establish conditions under which $(M_E) \equiv (M_G)$.
(e.g. $(M_E) \neq (M_G)$ in case $|| \cdot ||_E = || \cdot ||_{\ell^\infty}$, calibration example in [BOO])

Thank you for your attention!