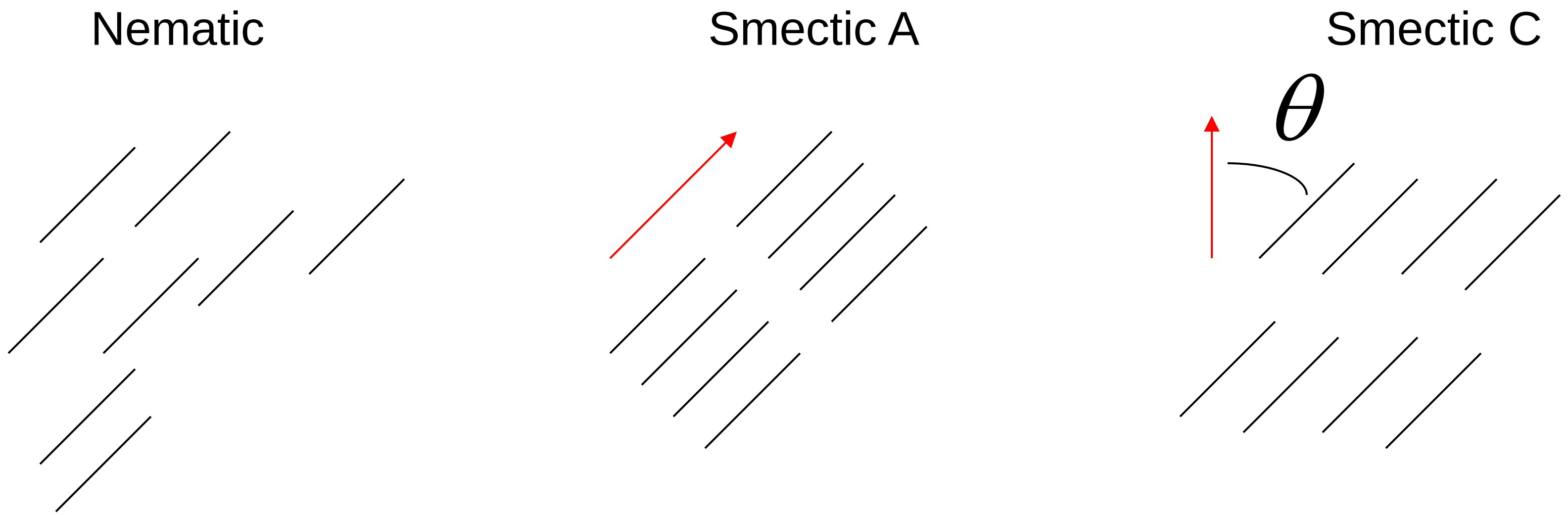
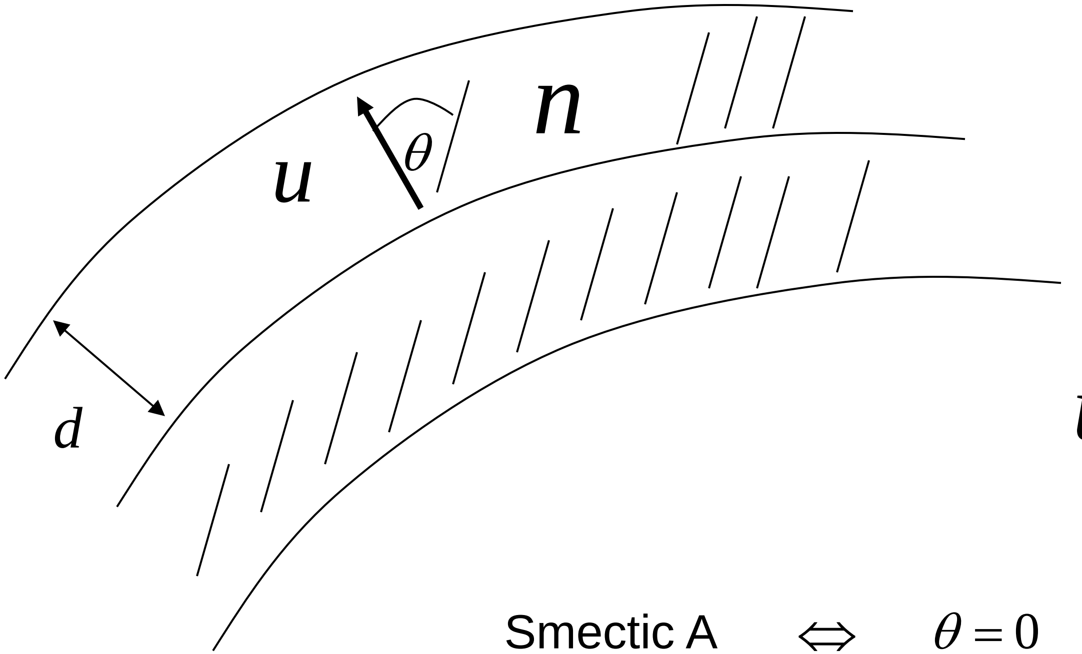
Chevron Structures in Liquid Crystal Cells

Lei Cheng, Lidia Mrad, Dan Phillips

Liquid Crystal Phases





$$q = \frac{2\pi}{d}$$

$$u = \nabla \omega$$

Smectic A $\Leftrightarrow$ $\theta = 0$ (de Gennes)

Smectic C $\Leftrightarrow$ $\theta \neq 0$ (Chen- Lubensky)

The level curves of ω represent layers.

$$\mathcal{F}_q = \int_\Omega (\frac{1}{q} f_s + f_n + f_e)$$

$$f_s = f_s(\psi, \mathbf{n}) \quad , \quad f_n = f_n(\mathbf{n}),$$

$$f_e = f_e(\psi, \mathbf{n})$$

Ω – material body

$\psi \colon \Omega \to \mathbb{C}$, $\quad \mathbf{n} \colon \Omega \to \mathbb{S}^2$

ψ is a complex order parameter , $\quad \mathbf{n}$ is the director field

$$\psi(x) = \rho(x) e^{iq\omega(x)}$$

$\rho = 0$ nematic, $\quad \rho \neq 0$ smectic

$f_n(\mathbf{n})-$ Frank-Oseen energy

$$f_{\mathbf{n}} = \frac{1}{2}K|\nabla\mathbf{n}|^2$$

The material is "chiral" however the domain Ω will be thin and due to this the specific elastic features are not significant.

$f_e(\mathbf{n}, \psi)-$ Electrostatic energy

Chirality induces a spontaneous polarization field, $\mathbf{P} = P_0\big(\nabla\omega \times \mathbf{n}\big)$.

If $\mathbf{E}$ is an applied electric field then

$$f_e = -\mathbf{E} \cdot \mathbf{P}$$

$f_s(\mathbf{n}, \psi)$– Smectic C energy density

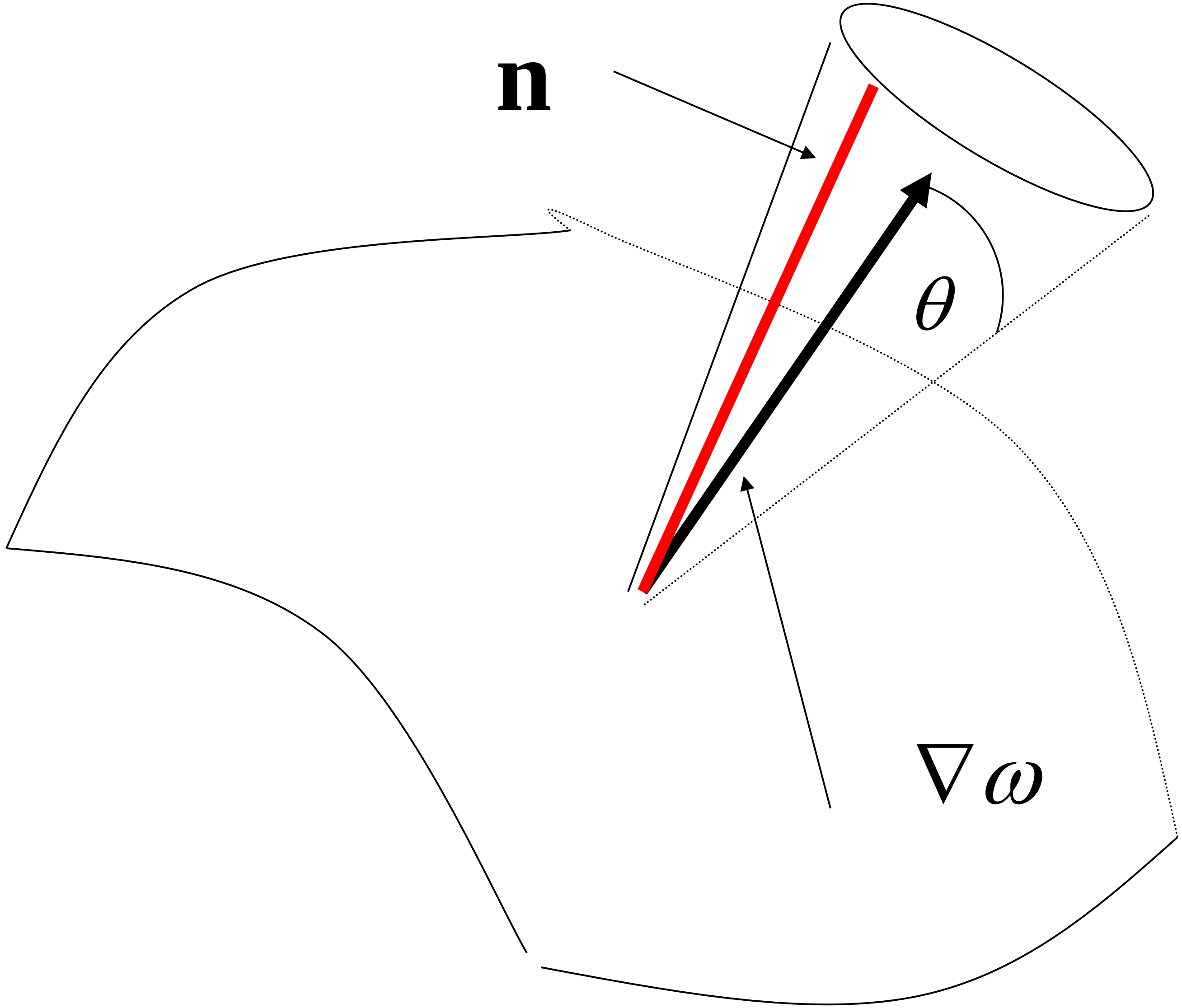
$$
\begin{aligned}
f_s &= \frac{a_\perp}{q^2}|D \cdot D_\perp \psi|^2 - c_\perp |D_\perp \psi|^2 \\
&+ \frac{a_\parallel}{q^2}|D \cdot D_\parallel \psi|^2 + c_\parallel |D_\parallel \psi|^2 \\
&+ qg(|\psi|^2 - 1)^2 \\
& \quad a_\perp, c_\parallel, c_\perp, a_\parallel > 0, \text{ where } \frac{a_\perp}{2c_\perp} = \sin^2\theta. \\
D &= \nabla - iq\cos\theta\mathbf{n}, \quad D_\parallel = (\mathbf{n} \cdot \nabla - iq\cos\theta)\mathbf{n} \\
D_\perp &= D - D_\parallel.
\end{aligned}
$$

Let $\psi = e^{iq\omega(\mathbf{x})}$, $\quad \mathbf{n} = \mathbf{n}(\mathbf{x})$.

$$\begin{aligned}
\frac{1}{q} f_s &= \frac{1}{q}\Big\{a_\perp\big(\text{div }(\nabla\omega - (\mathbf{n}\cdot\nabla\omega)\mathbf{n})\big)^2 + a_\|\big(\text{div }(((\mathbf{n}\cdot\nabla\omega) - \cos\theta)\mathbf{n})\big)^2\Big\} \\
&+ q\Big\{a_\perp(|\nabla_\perp\omega|^2 - \sin^2\theta)^2 \\
&+ a_\|((\nabla\omega\cdot\mathbf{n}) - \cos\theta)^4 + c_\|(\nabla\omega\cdot\mathbf{n} - \cos\theta)^2\Big\} + \text{ const}
\end{aligned}$$

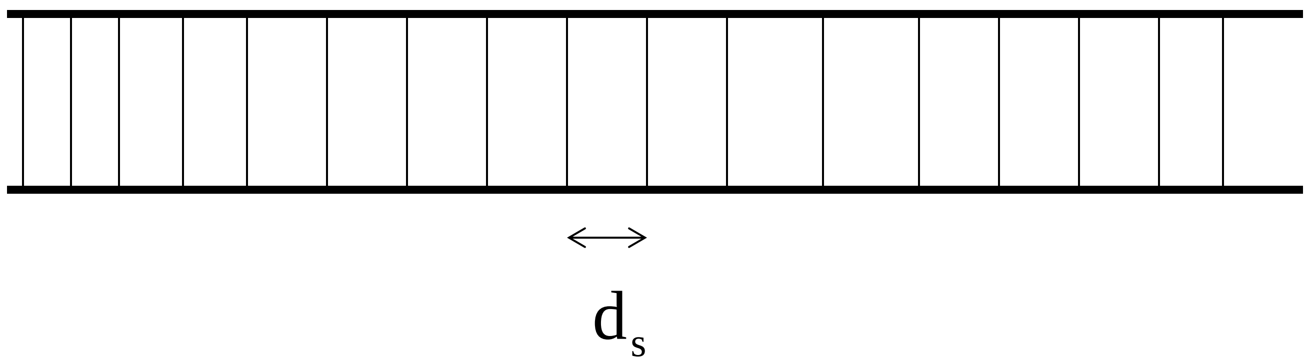
For q large we see:

- vanishing costs of specific curvatures for the smectic layers ($\omega = \text{const}$)
- $\nabla\omega\cdot\mathbf{n} \to \cos\theta$ (fixed tilt)
- $|\nabla\omega| \to 1$ (uniform spacing)



n
θ
∇ω

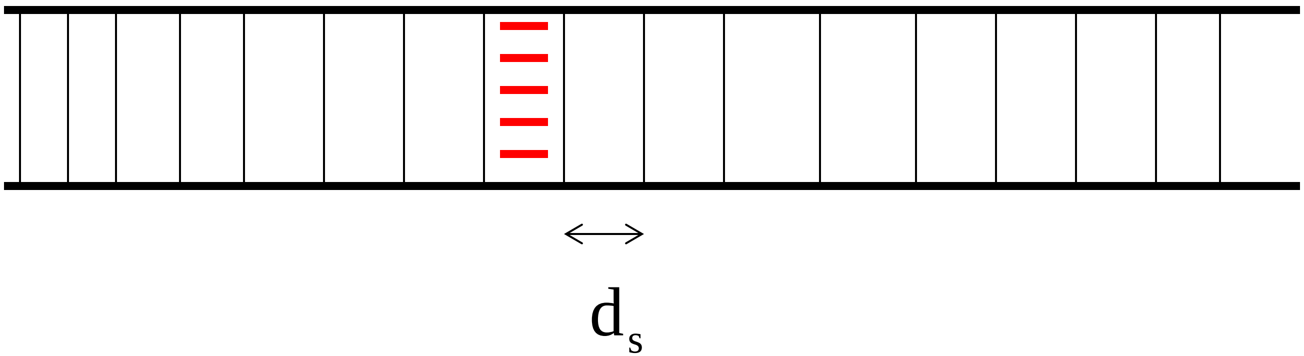
In surface-stabilized cells where the liquid crystals are confined between close glass plates with fixed boundary conditions.



The layer thickness on the boundary is given by d_s.

In surface-stabilized cells where the liquid crystals are confined between close glass plates with fixed boundary conditions.

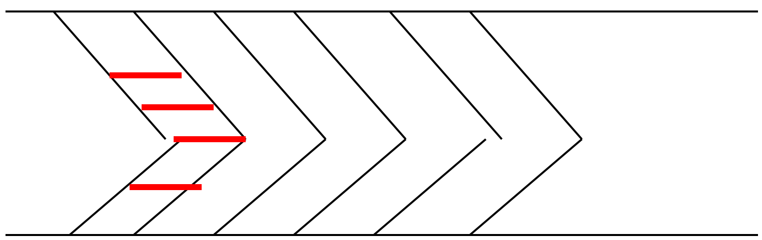
Bookshelf Geometry

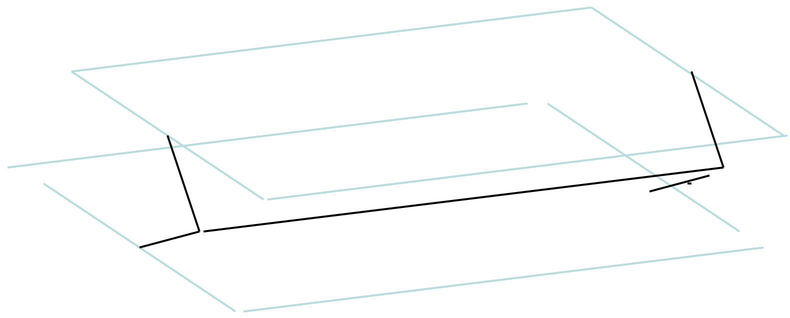


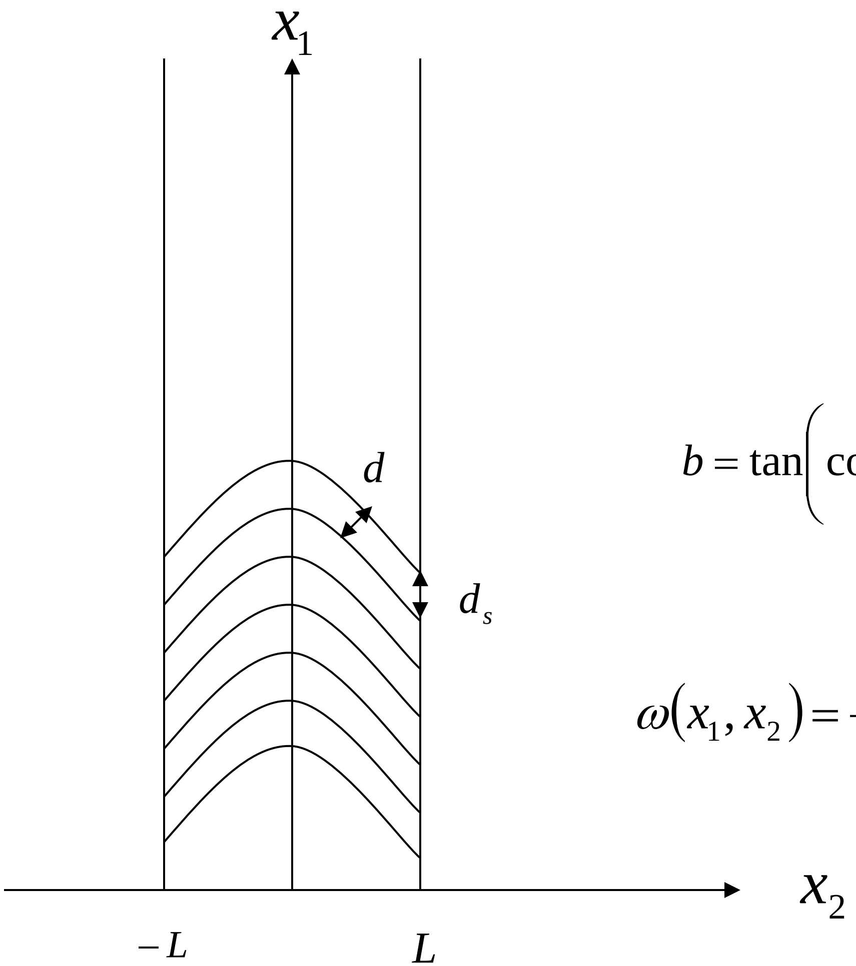
The layer thickness on the boundary is given by d_s.

We assume initially that the l.c. is in the smectic A phase.

- When the temperature is reduced the material enters the smectic C phase and the bookshelf structure deforms into V-shaped (chevron) layers.
- It is caused by two effects. The layers thin as the l.c. molecules tilt. The boundary layer spacing d_s does not change.
- It causes distortions in director pattern.

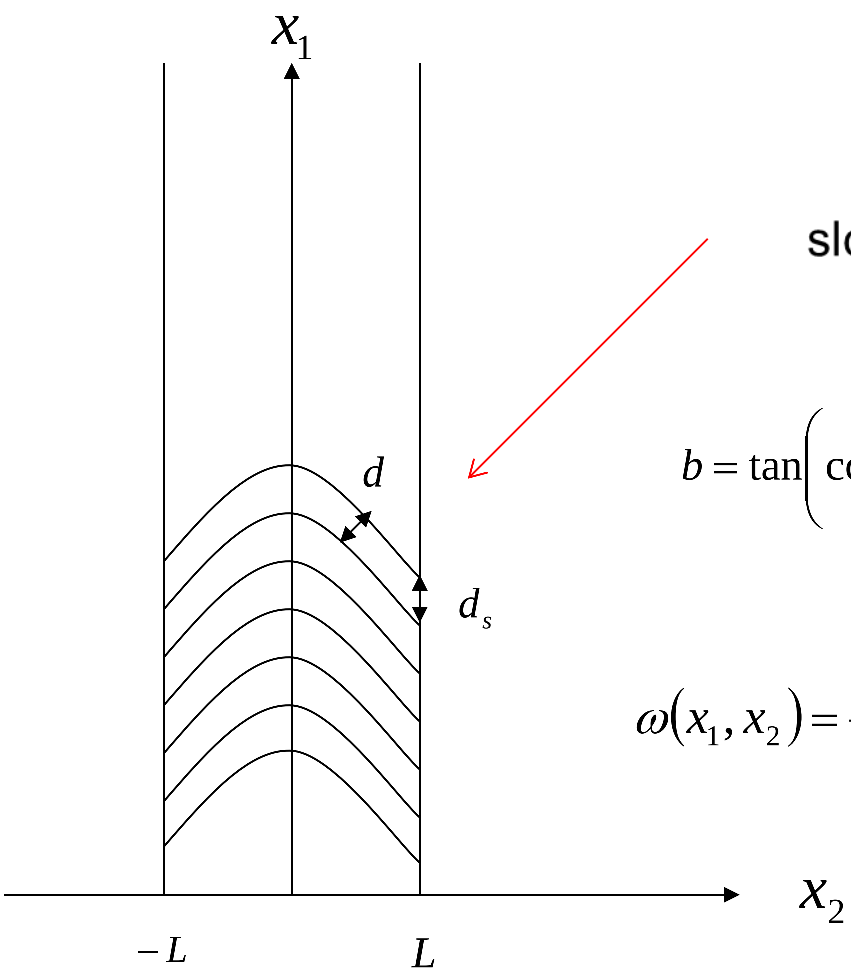






$$b = \tan\left(\cos^{-1} \frac{d}{d_s} \right)$$

$$\omega(x_1, x_2) = \frac{x_1 - g(x_2)}{\sqrt{1 + b^2}}$$



slope ≈ $\pm b$

$$b = \tan\left(\cos^{-1}\frac{d}{d_s}\right)$$

$$\omega(x_1, x_2) = \frac{x_1 - g(x_2)}{\sqrt{1+b^2}}$$

We consider a reduced setting such that:

- The domain is
$$\Omega = \{(x_1, x_2) | -L < x_2 < L\}.$$

- $\mathbf{n} = (n_1(x_2), n_2(x_2), n_3(x_2)),$

- Let $\omega(x_1, x_2) = \frac{x_1 - g(x_2)}{\sqrt{1+b^2}}$, where $g(x_2)$ is the layer displacement, and $b = tan(cos^{-1}\frac{d}{d_s})$.

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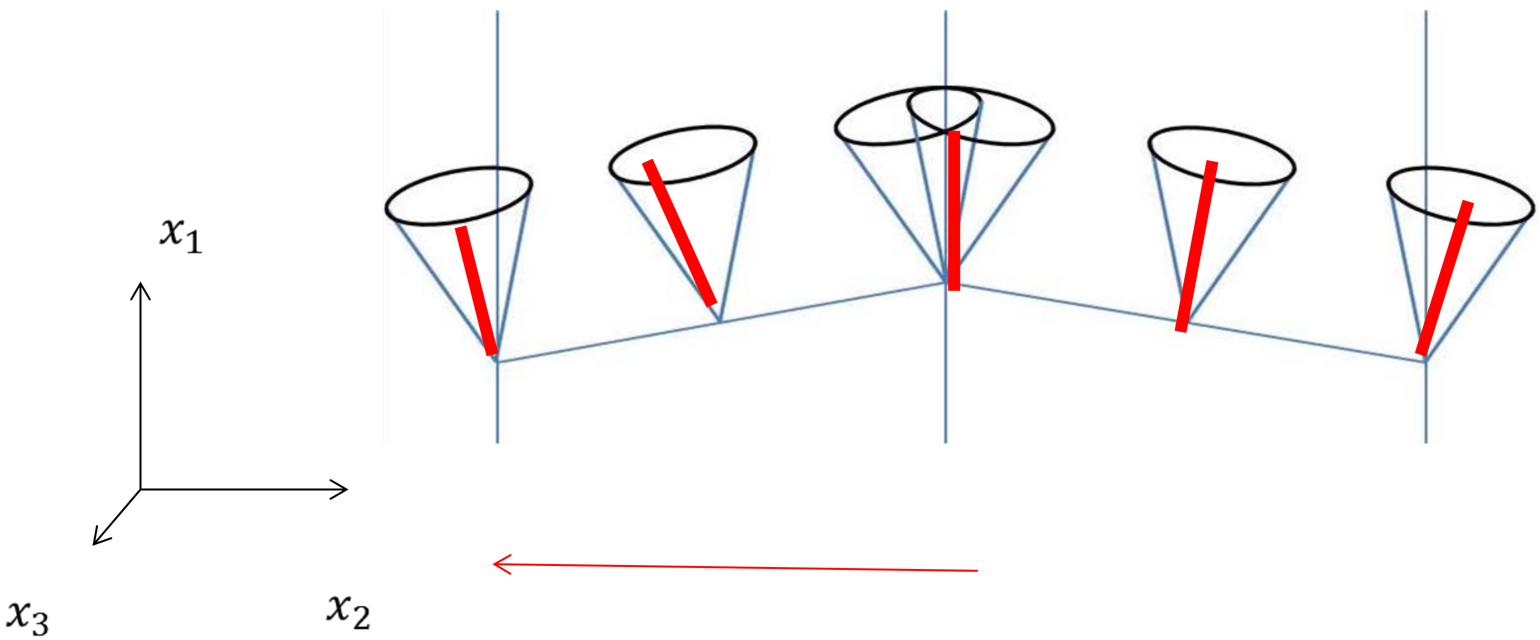
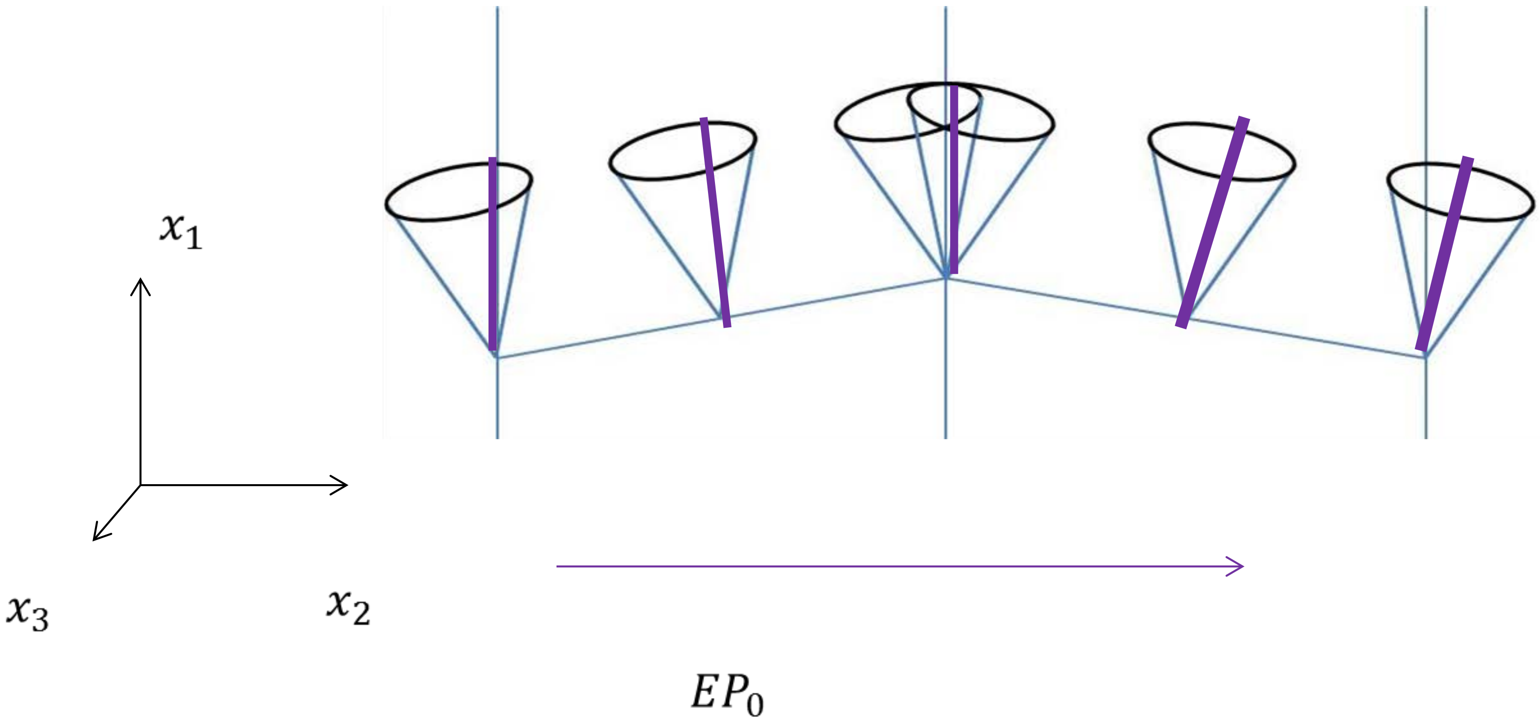
- $\mathbf{E} = (0, E, 0)$

$\Rightarrow$ that $-\mathbf{E} \cdot \mathbf{P} = \frac{P_0 E}{\sqrt{1+b^2}} n_3$.



EP_0



x_1
x_3
x_2
EP_0

Now the total energy can be simplified into

$$
\begin{aligned}
\mathcal{F}_q(,\mathbf{n},g') = \int_{-L}^{L} \Big\{ \frac{1}{q} \Big[& \frac{a_\perp}{1+b^2}\Big(\Big(-g' - n_1 n_2 + n_2^2 g'\Big)'\Big)^2 \\
& + \frac{a_\parallel}{1+b^2}\Big(\Big(n_1 n_2 - n_2^2 g' - n_2 \cos\theta\sqrt{1+b^2}\Big)'\Big)^2\Big] \\
& + q\Big[\frac{a_\perp}{(1+b^2)^2}\Big(1 + (g')^2 - (n_1 - n_2 g')^2 - \sin^2\theta(1+b^2)\Big)^2 \\
& + \frac{a_\parallel}{(1+b^2)^2}\Big(n_1 - n_2 g' - \cos\theta\sqrt{1+b^2}\Big)^4 \\
& + \frac{c_\parallel}{1+b^2}\Big(n_1 - n_2 g' - \cos\theta\sqrt{1+b^2}\Big)^2\Big] \\
& + \frac{1}{2}K\big(n_1'^2 + n_2'^2 + n_3'^2\big) \\
& + \frac{EP_0}{\sqrt{1+b^2}} n_3 \Big\} \, dx_2
\end{aligned}
$$

Our goal is to analyze the minimizers for $\mathcal{F}_q$ and their limiting behavior.

The admissible set:

$$\mathbb{X} = \{\, (g', \mathbf{n}) \,|\, g' \in \mathbf{L}^2(-L, L),\ g(-L) = g(L), \\ \mathbf{n} \in \mathbf{L}^2(-L, L),\ |\mathbf{n}| = 1 \,\}$$

- Can we find the function pairs $(g'_q, \mathbf{n_q})$ in the admissible set $\mathbb{X}$ that minimize the energy for $q > 0$?
- If yes, what do the minimizers look like?
- Is there a limiting problem as $q \to \infty$?

Establishing the $\Gamma-$Convergence result for $(\mathcal{F}_q)$

Theorem 1 (L.Cheng-D.P.) (2015) *For every $q > 0$, set*

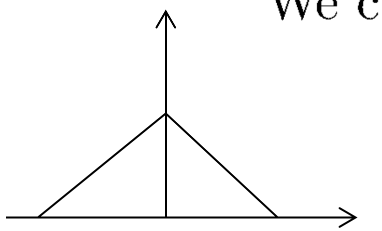
$$\mathcal{F}_q(g', \mathbf{n}) = \begin{cases} \int_{-L}^{L} F_q(g', \mathbf{n})\, \mathrm{d}x_2 & \text{if } (g', \mathbf{n}) \in \mathbb{X}, g', \mathbf{n} \in W^{1,2}(-L, L) \\ \infty & \text{elsewhere in } \mathbb{X} \end{cases}$$

Then as $q \to \infty$, the functionals $(\mathcal{F}_q)$ $\Gamma-$converge in $\mathbb{X}$ to

$$\mathcal{F}_\infty(g', \mathbf{n}) = \begin{cases} C_0 \|g'\|_{\mathsf{BV}} + \int_{-L}^{L} (\frac{1}{2} K |\mathbf{n}'|^2 + E P_0 n_3)\, \mathrm{d}x_2 & \text{if } (g', \mathbf{n}) \in \mathbb{X} \cap \mathbb{A} \\ \infty & \text{elsewhere in } \mathbb{X} \end{cases}$$

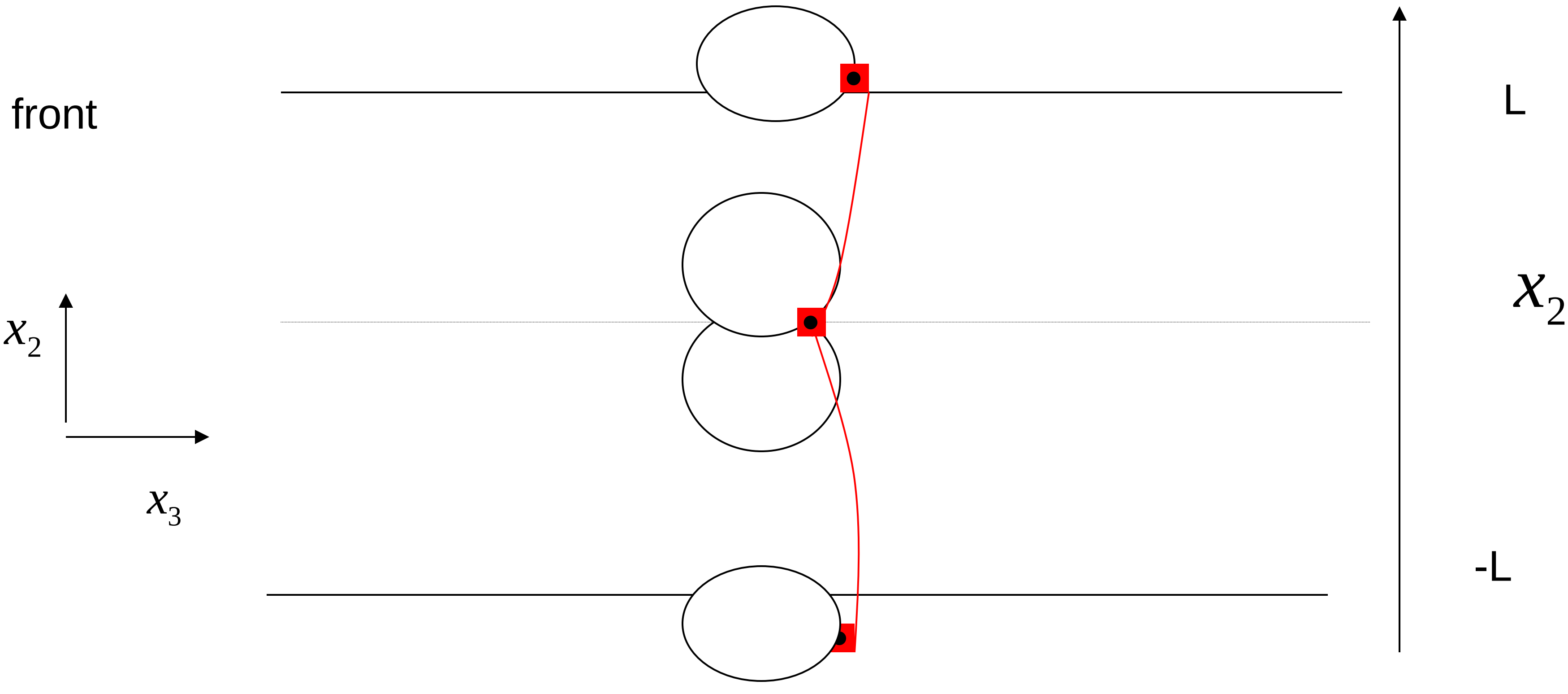
where $\mathbb{A} = \{(g', \mathbf{n}) | g' \in BV, n_2 = 0$ *at jumps of* g', $|g'| = b$,

$$n_1 - n_2 g' = \cos\theta \sqrt{1 + b^2} \;\; \text{on} \;\; (-L, L)\} \;\; \text{and} \;\; C_0 = \frac{4 a_\perp b^2}{3(1 + b^2)^{\frac{3}{2}}}$$



We can show that

$$g'(x) = \begin{cases} b & \text{if } x \in (-L, 0) \\ -b & \text{if } x \in (0, L) \end{cases}$$



- The reduced energy $\mathcal{F}_\infty$ does not have a path allowing switching from $E \to -E$ with finite energy. This means for large q it takes a very large E to induce switching for the gradient flow for $\mathcal{F}_q$.

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- To get a theory with a reduced energy that allows switching with a finite energy barrier one needs to work with f_s replacing the ansatz $\psi(x_1, x_2) = e^{\frac{q(ix_1 - g(x_2))}{\sqrt{1+b^2}}}$ with $\psi(x_1, x_2) = e^{\frac{iqx_1}{\sqrt{1+b^2}}}\tilde{\psi}(x_2)$ such that $\tilde{\psi}(x_2) \in \mathbb{C}$ and can vanish.

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Mathematical Model

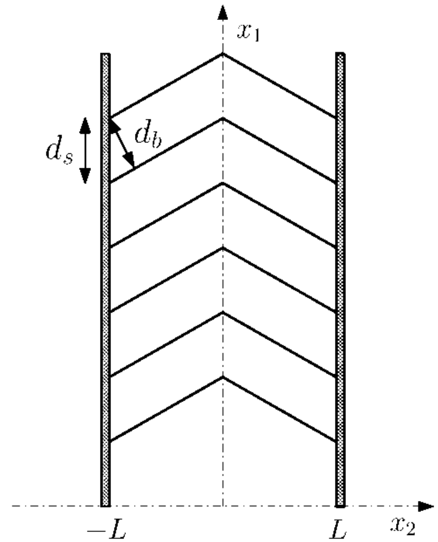
$$\frac{1}{q} f_s(\mathbf{n}, \psi) = G(\mathbf{n}, \psi) + H(|\psi|)$$

$$G(\mathbf{n}, \psi) = \frac{a_\perp}{q^3} \left| D \cdot D_\perp \psi + \frac{c_\perp q^2}{2a_\perp} \psi \right|^2 + \frac{a_\parallel}{q^3} \left| D \cdot D_\parallel \psi \right|^2 + \frac{c_\parallel}{q} \left| D_\parallel \psi \right|^2$$

$$H(|\psi|) = g(|\psi|^2 - 1)^2 + |\nabla |\psi|^2|^2 + \frac{1}{q^2} |\nabla^2 |\psi|^2|^2 + \frac{1}{q^6} |\nabla^3 |\psi|^2|^2$$

$G(\mathbf{n}, \psi)$: describes how $\mathbf{n}$ relates to the layers

$H(|\psi|)$: describes the energetic cost leaving the smectic phase



We consider a 2-D cross-section of the cell.

Periodicity: Reduce to 1-D model as first suggested by Sluckin et al.,
$\widetilde{\psi}(x_1, x_2) = \rho e^{i \frac{q[x_1 - g(x_2)]}{\sqrt{1+b^2}}}$.

Main difference: We reduce the model but keep the 1-D complex-valued parameter in general form: $e^{\frac{iqx_1}{\sqrt{1+b^2}}}\psi(x_2)$, allowing for $|\psi| = 0$.

$$\partial_t(\psi, \mathbf{n}) = -\delta \mathcal{F}_q(\psi, \mathbf{n})$$

where $(\psi(x_2, 0), \mathbf{n}(x_2, 0))$ is close to $(e^{\frac{-iqg(x_2)}{\sqrt{1+b^2}}}, \mathbf{n}_0(x_2))$

such that

$(g', \mathbf{n}_0) \in \mathbb{A} = \{(g', \mathbf{n}) | g' \in BV, n_2 = 0 \text{ at jumps of } g', \ |g'|{=}b,$

$$n_1 - n_2 g' = \cos\theta\sqrt{1+b^2} \text{ on } (-L, L)\}$$

Existence and uniqueness results for continuous L^2- gradient flows

$(\psi(x_2,t),\mathbf{n}(x_2,t))$ for:

$$\mathcal{F}_q(\psi,\mathbf{n}) = \int_{-L}^{L}\Big\{\frac{a_\perp}{q}|[\frac{\psi'}{q} - (\frac{i}{\sqrt{1+b^2}}n_1\psi + n_2\frac{\psi'}{q})n_2]' - \frac{q}{1+b^2}\psi$$

$$+\frac{q}{1+b^2}n_1^2\psi - \frac{i}{\sqrt{1+b^2}}n_1n_2\psi' + qsin\theta\psi|^2$$

$$+\frac{a_\|}{q}|(\frac{n_1}{\sqrt{1+b^2}} - cos\theta)(-\frac{q}{\sqrt{1+b^2}}n_1\psi + in_2\psi' + qcos\theta\psi)$$

$$+[(\frac{i}{\sqrt{1+b^2}}n_1\psi + n_2\frac{\psi'}{q} - icos\theta\psi)n_2]'|^2 + c_\||\frac{i}{\sqrt{1+b^2}}n_1\psi + n_2\psi' + qcos\theta\psi|^2$$

$$+k(|\psi|^2-1)^2 + h(|\psi|^{2'})^2 + \frac{\alpha}{q^2}(|\psi|^{2''})^2 + \frac{\beta}{q^6}(|\psi|^{2'''})^2$$

$$+\frac{K}{2}|\mathbf{n}'|^2 + \frac{PE}{\sqrt{1+b^2}}|\psi|^2n_3\Big\}\,dx_2$$

Statics

- We prove coercivity and lower semi-continuity of the energy $\mathcal{F}_q(\mathbf{n}, \psi)$.

- We construct well-prepared initial data, ind. of q. (A specific $(\mathbf{n}, \psi) = (\mathbf{n}_q, \psi_q)$ for which $\mathcal{F}_q(\mathbf{n}, \psi) \leq C$ where C is ind. of q.)

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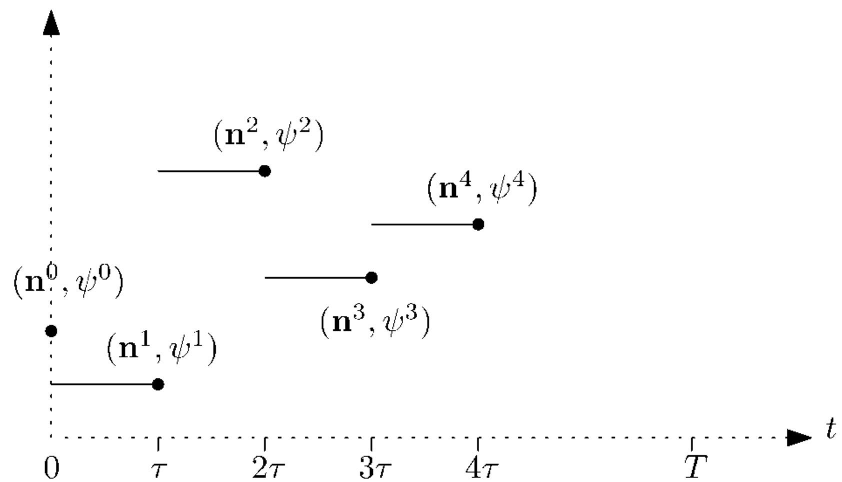
- This is a large class of initial conditions $\{(\mathbf{n}_q, \psi_q)\}$ that can approximate states with finite numbers of chevrons.

Dynamics: Gradient Flow

To study the dynamic behavior, we construct a discrete-in-time gradient flow following *Rothe's method*.

$$J^0(\mathbf{n},\psi) = \int_{-L}^{L} \left\{ \frac{|\mathbf{n}-\mathbf{n}^0|^2}{2\tau} + \frac{|\psi-\psi^0|^2}{2\tau} \right\} dx + \mathcal{F}_q(\mathbf{n},\psi)$$

- Minimize J^0 over $[0,\tau]$ with initial conditions $(\mathbf{n}^0,\psi^0)$.
- Get minimizer $(\mathbf{n}^1,\psi^1)$.
- Use $(\mathbf{n}^1,\psi^1)$ as an initial condition for $[\tau,2\tau]$, minimize J^1 and iterate.
- Get a sequence of minimizers $\{(\mathbf{n}^m,\psi^m)\}$.
- Connect these minimizers in a piecewise costant fashion to get the discretized minimizers $\{(\mathbf{n}^\tau,\psi^\tau)\}$ over $[0,T]$.



We consider a 2-D cross-section of the cell with a piecewise constant in time solution.

Energy Dissipation:

$$\frac{1}{2}\sum_{k=1}^{m}\tau(||\delta_\tau \mathbf{n}^k||_2^2 + ||\delta_\tau \psi^k||_2^2) + \mathcal{F}_q(\mathbf{n}^m, \psi^m) \le \mathcal{F}_q(\mathbf{n}^0, \psi^0) \text{ for } 1 \le m \le M \text{ where } M\tau < T.$$

With well-prepared initial data, energy at any later time is controlled (independent of q).

smectic elastic term

$$\mathcal{F}_q(\psi,\mathbf{n}) = \int_{-L}^{L}\Big\{\frac{a_\perp}{q}\big|[\frac{\psi'}{q} - (\frac{i}{\sqrt{1+b^2}}n_1\psi + n_2\frac{\psi'}{q})n_2]' - \frac{q}{1+b^2}\psi$$

$$+\frac{q}{1+b^2}n_1^2\psi - \frac{i}{\sqrt{1+b^2}}n_1n_2\psi' + qsin\theta\psi\big|^2$$

$$+\frac{a_\parallel}{q}\big|(\frac{n_1}{\sqrt{1+b^2}} - cos\theta)(-\frac{q}{\sqrt{1+b^2}}n_1\psi + in_2\psi' + qcos\theta\psi)$$

$$+[(\frac{i}{\sqrt{1+b^2}}n_1\psi + n_2\frac{\psi'}{q} - icos\theta\psi)n_2]'\big|^2 + c_\parallel\big|\frac{i}{\sqrt{1+b^2}}n_1\psi + n_2\psi' + qcos\theta\psi\big|^2$$

$$+k(|\psi|^2-1) + h(|\psi|^{2'})^2 + \frac{\alpha}{q^2}(|\psi|^{2''})^2 + \frac{\beta}{q^6}(|\psi|^{2'''})^2$$

$$+\frac{K}{2}|\mathbf{n}'|^2 + \frac{PE}{\sqrt{1+b^2}}|\psi|^2n_3\Big\}dx_2$$

term determining the bulk phase

$$\mathcal{F}_q(\psi,\mathbf{n}) = \int_{-L}^{L}\Big\{\frac{a_\perp}{q}|[\frac{\psi'}{q} - (\frac{i}{\sqrt{1+b^2}}n_1\psi + n_2\frac{\psi'}{q})n_2]' - \frac{q}{1+b^2}\psi$$

$$+\frac{q}{1+b^2}n_1^2\psi - \frac{i}{\sqrt{1+b^2}}n_1 n_2\psi' + qsin\theta\psi|^2$$

$$+\frac{a_\parallel}{q}|(\frac{n_1}{\sqrt{1+b^2}} - cos\theta)(-\frac{q}{\sqrt{1+b^2}}n_1\psi + in_2\psi' + qcos\theta\psi)$$

$$+[(\frac{i}{\sqrt{1+b^2}}n_1\psi + n_2\frac{\psi'}{q} - icos\theta\psi)n_2]'|^2 + c_\parallel|\frac{i}{\sqrt{1+b^2}}n_1\psi + n_2\psi' + qcos\theta\psi|^2$$

$$+k(|\psi|^2-1)^2 + h(|\psi|^{2\prime})^2 + \frac{\alpha}{q^2}(|\psi|^{2\prime\prime})^2 + \frac{\beta}{q^6}(|\psi|^{2\prime\prime\prime})^2$$

$$+\frac{K}{2}|\mathbf{n}'|^2 + \frac{PE}{\sqrt{1+b^2}}|\psi|^2 n_3\Big\}dx_2$$

electrostatic forcing term

$$\mathcal{F}_q(\psi,\mathbf{n}) = \int_{-L}^{L}\Big\{\frac{a_\perp}{q}|[\frac{\psi'}{q} - (\frac{i}{\sqrt{1+b^2}}n_1\psi + n_2\frac{\psi'}{q})n_2]' - \frac{q}{1+b^2}\psi$$

$$+\frac{q}{1+b^2}n_1^2\psi - \frac{i}{\sqrt{1+b^2}}n_1n_2\psi' + qsin\theta\psi|^2$$

$$+\frac{a_\parallel}{q}|(\frac{n_1}{\sqrt{1+b^2}} - cos\theta)(-\frac{q}{\sqrt{1+b^2}}n_1\psi + in_2\psi' + qcos\theta\psi)$$

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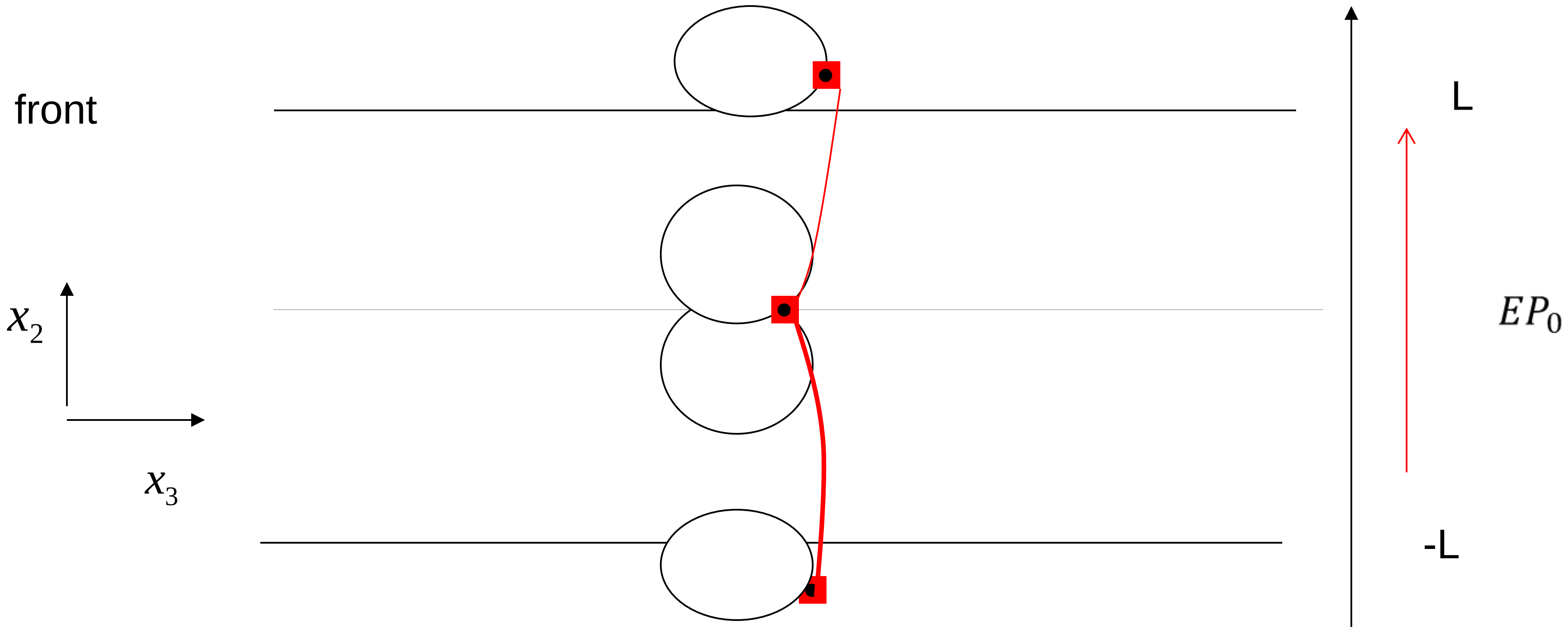
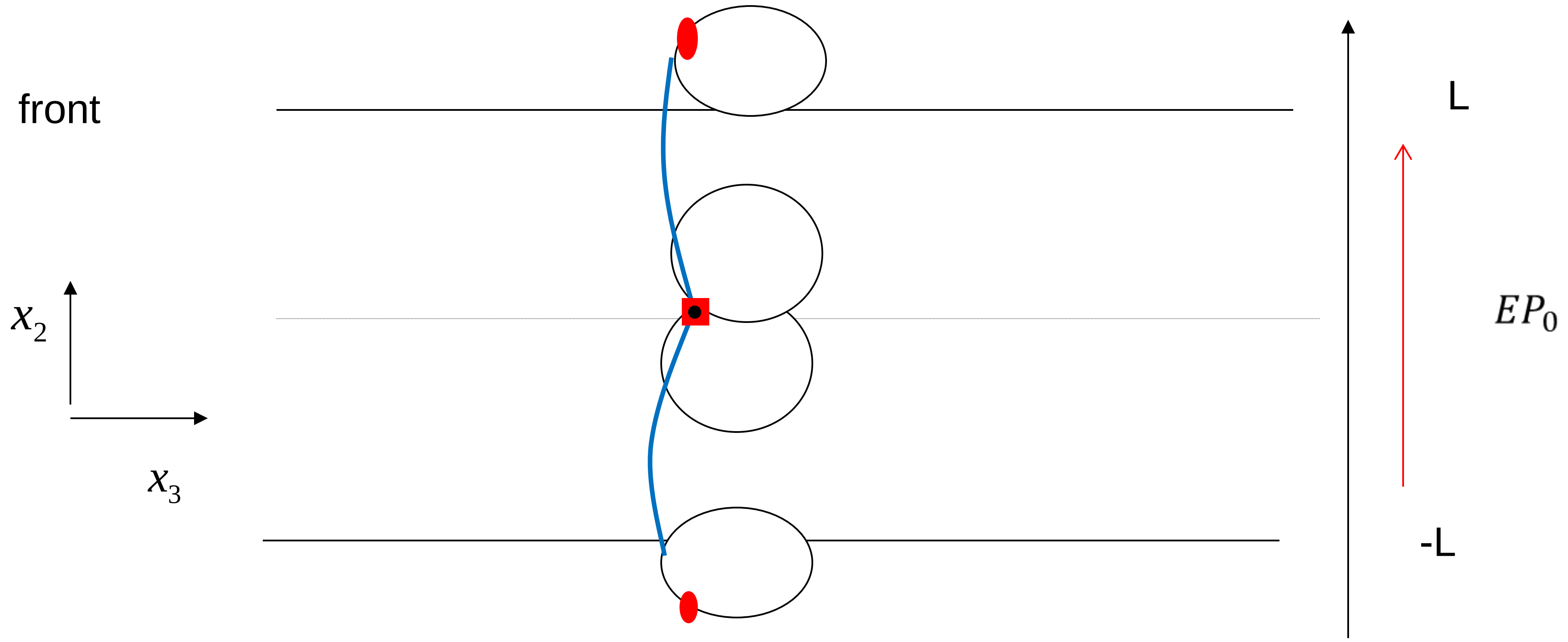
$$+k(|\psi|^2-1) + h(|\psi|^{2'})^2 + \frac{\alpha}{q^2}(|\psi|^{2''})^2 + \frac{\beta}{q^6}(|\psi|^{2'''})^2$$

$$+\frac{K}{2}|\mathbf{n}'|^2 + \frac{PE}{\sqrt{1+b^2}}|\psi|^2n_3\Big\}dx_2$$

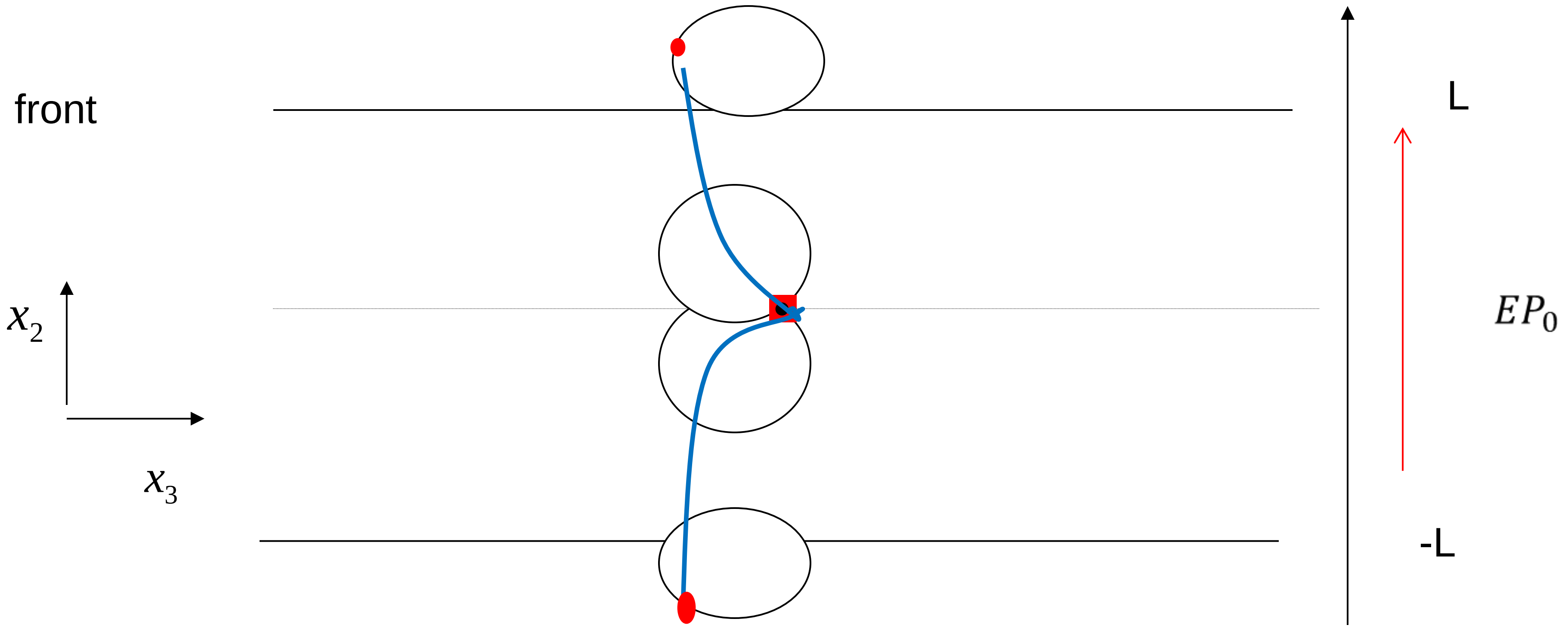
The idea is to have a flow that can go from here at t=0



to here for $t >> 0$.



As opposed to tending to this...



Idea of Proof:

We need to prove the convergence of the discrete gradient flow.

$$\int_0^T \int_{-L}^{L} [\delta_\tau \mathbf{n} + U_1^\tau]\, \mathbf{u} + U_2^\tau \mathbf{u}'\, dxdt = 0$$

To get an idea of the *nonlinearity* of U_1^τ:

$$\begin{aligned} U_1^\tau &= 2\Re\left\{ \frac{a_\perp}{q} \left(\frac{i}{\sqrt{b^2+1}} n_2^{\tau\prime} \overline{\psi^\tau} + \ldots \right) \left(\frac{\psi^{\tau\prime\prime}}{q} + \ldots \right) \right\} + \ldots \\ &= C n_2^{\tau\prime} \Im \left\{ \overline{\psi^\tau} \psi^{\tau\prime\prime} \right\} + \ldots \end{aligned}$$

It is enough to prove:
$U_1^\tau \to U_1$ in $L^1(\Omega_T)$
U_1^τ is bounded in $L^2(\Omega_T)$

$$\Rightarrow U_1^\tau \underset{\tau \to 0}{\rightharpoonup} U_1 \text{ in } L^2(\Omega_T).$$

- We get this convergence with two types of estimates.
- Higher order estimates in x:

We can show there are constants $q_0(C)$ and $M(C,q)$ so that

$$\int_0^T \int_{-L}^{L} |\mathbf{n}^{\tau\prime\prime}|^2 + |\psi^{\tau\prime\prime\prime}|^2 + ||\psi^{\tau}|^{2\,(5)}|^2 \, dxdt \leq M(C,q), \quad \text{uniformly in } \tau > 0$$

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- An estimate on $|\psi'|$.

- This is due to a Modica-Mortola type estimate:

  There is a $K(C)$ so that if $\mathcal{F}_q(\mathbf{n}, \psi) \leq C$ and $1 \leq q$ then $\frac{|\psi'|}{q} \leq K$.

- We need this to show the energy is coercive.

- In the static problem we assumed the ansatz $\psi(x) = e^{iqg(x)}$ and showed $|g'(x)| \leq K$.

This came from:

$$\int_{-L}^{L} \left\{ \frac{1}{q} (g''(x))^2 + qC_0 \left[\left((g'(x))^2 - b^2 \right)^+ \right]^2 \right\} dx \le C_1 \mathcal{F}_q + C_2 \le C_3$$

$\Rightarrow$

$$|g'(x)| \le K$$

- For this case note that since $\mathcal{F}(\mathbf{n},\psi) \le \mathcal{F}(\mathbf{n}^0,\psi^0)$ we have

$$\int_{-L}^{L} ((|\psi|^4 + (|\psi|^{2\prime})^2 + \frac{1}{q^2}(|\psi|^{2\prime\prime})^2 + \frac{1}{q^6}(|\psi|^{2\prime\prime\prime})^2)\,dx \le C.$$

- It follows that $|\frac{|\psi|^{2\prime\prime}}{2q^2}| + \frac{b^2}{1+b^2}|\psi|^2 \le M$ where C and M are independent of q for $q \ge 1$.

- If $(\alpha,\beta) \subset [-L,L]$ is a maximal interval on which $|\psi| > 0$ we get

$$\int_{\alpha}^{\beta} \frac{1}{q}\left|\Im\{\frac{\psi''\overline{\psi}}{q}\}\right|^2 |\psi|^{-2} + q\left|\Re\{\frac{\psi''\overline{\psi}}{q^2}\} + \frac{b^2}{1+b^2}|\psi|^2\right|^2 |\psi|^{-2}\,dx \le C$$

Using the fact that the initial energy is bounded, and after carrying out some algebraic manipulations on the energy, we get

- From these inequalities we have a Modica-Mortola type estimate,

$$\int_{\alpha}^{\beta} \left| \left(\Re\{\frac{\psi'\overline{\psi}}{q|\psi|}\} \right)' \left[\left(\Re\{\frac{\psi'\overline{\psi}}{q|\psi|}\} \right)^2 - 2M \right]^+ \right| dx \leq C$$

- If Φ is such that $\Phi'(y) = [y^2 - 2M]^+$, then

$$\underset{(\alpha,\beta)}{osc} \ \Phi \left(\frac{|\psi|'}{q} \right) = \underset{(\alpha,\beta)}{osc} \ \Phi \left(\Re\{\frac{\psi'\overline{\psi}}{q|\psi|}\} \right) \leq C.$$

- If $|\psi(\alpha)| = |\psi(\beta)| = 0$ it follows that $|\psi(x)|' = 0$ for some $x \in (\alpha, \beta)$.
- If either $\alpha = -L$ or $\beta = L$ then it follows from the boundary conditions that $|\psi|' = 0$ at that point.
- In either case it follows that $|\frac{|\psi|'}{q}|$ is uniformly bounded independent of q and the interval (α, β).

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- A similar reasoning can be applied to the imaginary part $\frac{1}{q}\Im\{\frac{\psi'\overline{\psi}}{q|\psi|}\}$.
- $[-L, L] \setminus \{\bigcup_j [\alpha_j, \beta_j]\}$ are accumulation points for $\{\psi = 0\}$ so $\psi' = 0$ on this set.

- How does this relate to prior work?
- Čopič etal. consider

$$\mathcal{F}_q(\mathbf{n}, \psi) = \int_{-L}^{L} \{f_s(\mathbf{n}, \psi) + f_{\mathbf{n}}(\mathbf{n}) + f_e(n_3)\} \, dx,$$

with q fixed and choose sufficiently small elasticity constants for f_s in their simulations so that the three terms are of the same order.

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- We use q as a parameter

$$\mathcal{F}_q(\mathbf{n},\psi) = \int_{-L}^{L} \left\{\frac{1}{q} f_s(\mathbf{n},\psi) + f_{\mathbf{n}}(\mathbf{n}) + f_e(n_3)\right\} dx,$$

where

$$q = \frac{1}{\text{layer thickness}} \to \infty.$$

With this weighting the three energies have the same order for large q.

Results

- existence
- uniqueness (independent of the choice of minimizers as well as the particular discretization used)
- a simple picture when the wave number q is a sufficiently large constant.

- How can we use it?
- Characterization of and the dynamics for the limiting problem i.e. when $q \to \infty$.
- We now have a set-up allowing switching at the chevron tip.
- We expect to identify regions of melting
  around the chevron tip, where $\mathbf{n}$ decouples from the cone
  and switches continuously.