t-stack sortable permutations and log-concavity

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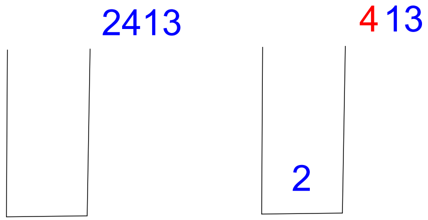
Stack sorting

Let $p = 2413$. Let us stack sort p.



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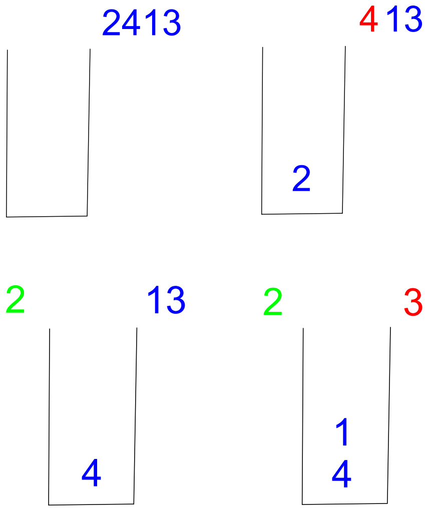

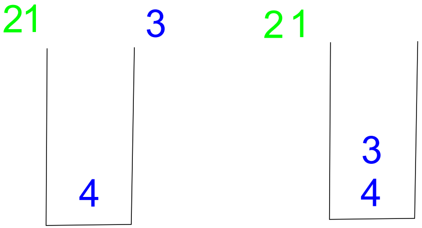
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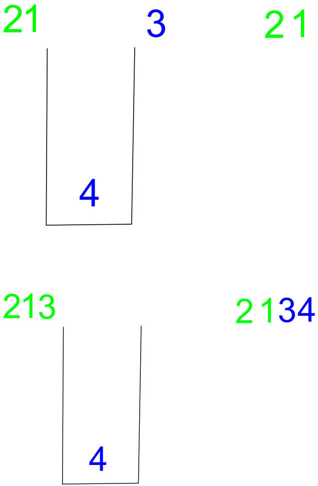
21
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Equivalent definitions

Let $p = LnR$, where L and R denote the strings on the left and on the right of the maximal entry n.

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Then

$$s(p) = s(L)s(R)n,$$

and this recursively defines the stack sorting operation.

Decreasing binary trees

In the tree $T(p)$ of the permutation $p = LnR$, the root has label n, the entries of L are in the left subtree, and the entries of R are in the right subtree. These subtrees are defined recursively by the same rule.

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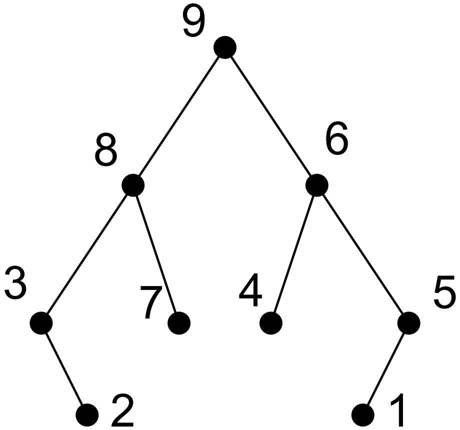

Figure: The tree $T(p)$ for $p = 328794615$.

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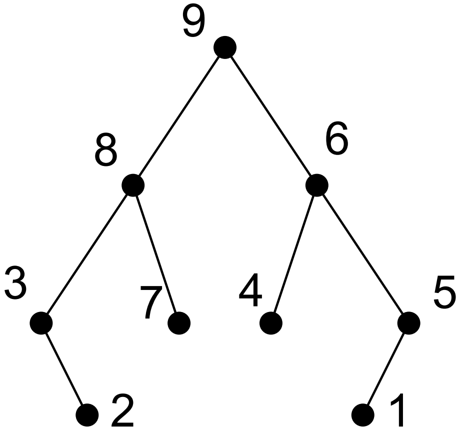

Figure: Here $s(p) = 237841569$.

Stack sortable permutations

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So, the number of stack sortable permutations of length n is the nth Catalan number, $\binom{2n}{n}/(n+1)$.

Descents

The number of stack sortable permutations of length n with $k-1$ descents is the Narayana number

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In particular, for fixed n, the sequence of stack sortable permutations of length n with k descents is symmetric and unimodal.

t-stack sortable permutations

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If $t > 1$, then t-stack sortability is *not* a monotone property.

Let $W_t(n)$ be the number of t-stack sortable permutations of length n, and let $W_t(n, k)$ be the number of such permutations with k descents.

When $t = 2$

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and

$$W_2(n,k) = \frac{(n+k)!(2n-k-1)!}{(k+1)!(n-k)!(2k+1)!(2n-2k-1)!}.$$

Lattice paths

The number of lattice paths with steps $(0, 1)$, $(1, 0)$ and $(-1, -1)$ that start and end at $(0, 0)$, use $3n$ steps, and never leave the first quadrant is equal to $2^{2n-1}W_2(n)$.

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A direct proof (one that does not resort to planar maps) is not known.

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For the purposes of generalizing to higher values of t, a simple argument showing that

$$W_2(n) < \binom{3n}{n}$$

would be more useful.

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My conjecture is that

$$W_t(n) < \binom{(t+1)n}{n}.$$

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$$\sqrt[n]{W_4(n)} \leq 21.97225.$$

Descents again

Theorem

(B, 2004) Let $W_t(n,k)$ be the number of t-stack sortable permutations of length n. Then for all fixed n and t, the sequence

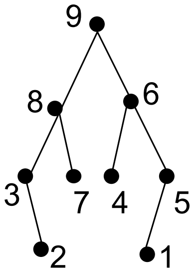
$$W_t(n,0), W_t(n,1), \cdots, W_t(n,n-1)$$

is symmetric and unimodal.

A different proof was given by Petter Brändén in 2008.

Idea of proof of symmetry

In $T(p)$, find the vertices that have *exactly one* child, and change the direction of the edge connecting that vertex to that child.



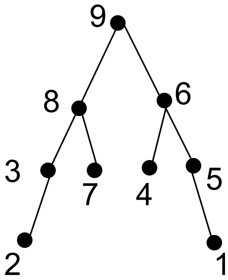


Figure: Turning $p = 328794615$ into $d(p) = 238794651$.

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Hence d turns a t-stack sortable permutation with k ascents into a t-stack sortable permutation with k descents.

Idea of proof of unimodality

We use the reflection principle. Let us say that $T(p)$ has $k < (n-1)/2$ right edges.

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Consider $T(p)$ as a poset, then find its lexicographically first ideal that contains one less right edges than left edges.

Now apply d to that ideal. The result is a tree with one more right edges. This injectively proves that $W_t(n,k) \leq W_t(n,k+1)$.

Real roots

Conjecture

Then for all fixed n and t, the polynomial

$$\sum_{k=0}^{n-1} W_t(n,k)x^k$$

has real roots only.
In particular, the sequence

$$W_t(n,0), W_t(n,1), \cdots, W_t(n,n-1)$$

is log-concave.

Special cases

For $t = 1$ and $t = 2$, log-concavity is routine to prove because of the explicit formulae known for the numbers $W_t(n, k)$.

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The real root property is not obvious, but is known to be true, by the work of Brenti and Brändén.

If $t = n - 1$, then all permutations of length n are t-stack sortable, so the numbers $W_t(n, k)$ are the well-known *Eulerian numbers*. So their generating polynomial is an Eulerian polynomial, and hence, it has real roots only.

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If $t = n - 2$, then the t-stack sortable permutations are all permutations of length n that *do not* end in $\cdots n1$. Real-rootedness is not obvious, but is known to be true, by a result of Brändén.

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The conjecture is open for all values of $t \in [3, n - 3]$.

Another log-concavity conjecture

Conjecture

For all n, the sequence $W_1(n), W_2(n), W_3(n), \cdots$ *is log-concave.*