Generated Jacobian equations: from Geometric Optics to Economics

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9 April 2017–14 April 2017

1 Overview of the Field

The field of generated Jacobian equations (GJE) is a nascent field, motivated on one hand by questions in geometric optics and partial differential equations, and on the other by structures that arise naturally in economics. It may be cast as an outgrowth of the theory of optimal transport (OT), which has seen a lot of development and progress over the last couple of decades.

As may be gleaned from the fields mentioned above, the span of areas involving the analysis of generated Jacobian equations is vast. This ubiquity of GJE and OT reflects the fact that they are concerned with the study of the following common situation: one has mappings (alternatively "changes of variables", or "matchings") with a prescribed action on volume and which satisfy an optimality condition. Such situations include measure preserving mappings that minimize a cost functional, optimal matchings between two or more populations, couplings of random variables that minimize covariance, ray tracing maps from one reflecting surface to another, to name a few.

As we are dealing with mappings with a prescribed action on given measures, we are led naturally to the notion of a **prescribed Jacobian equation**. This is an equation where the unknown is a map T (sometimes, but not always a diffeomorphism) between two spaces X and Y (frequently Riemannian manifolds or subsets of Euclidean space), and the distortion of the volume by this map is **prescribed**. This leads to equations of the form

$$\det DT(x) = \psi(x, T(x)).$$

In this generality, this is a nonlinear system of PDEs, however, we are interested in a more special situation. If T is a gradient map, that is if $T(x) = \nabla u(x)$ for some scalar valued potential function u, the equation turns into the following second order scalar PDE, known as the **Monge-Ampère** equation:

$$\det D^2 u(x) = \psi(x, \nabla u(x)).$$

The most studied class of solutions to this equation is the class of convex u where ψ is nonnegative, in which case the equation is degenerate elliptic. In the study of more general prescribed Jacobian equations, there is an analogue of "mapping arising from a convex potential" through the introduction of what is known as a **generating function** G. This refers to a real valued function,

$$G : (x, y, z) \in X \times Y \times \mathbb{R} \mapsto \mathbb{R},$$

which determines a **duality structure** between the spaces X and Y. This generalizes the notion of duality between a vector space and its dual, which corresponds to the choice of

$$G(x, y, z) = -\langle x, y\rangle + z,$$

(here $\langle \cdot, \cdot\rangle$ is the duality pairing between a vector and covector). The theory of GJE is concerned with prescribed Jacobian equations that arise from a generating function. Concretely, they involve an unknown scalar function $u : \Omega \mapsto \mathbb{R}$ solving a nonlinear PDE of the form

$$\det D(T_u(x)) = \psi(x, u(x), \nabla u(x)), \;\; T_u(x) = T(x, u(x), Du(x)),$$

where T_u is a generalization of the gradient map from the classical Monge-Ampère case above, which is determined by solving the following system of equations in T and Z:

$$D_x G(x, T(x, u, \bar{p}), Z(x, u, \bar{p})) = \bar{p}, \; G(x, T(x, u, \bar{p}), Z(x, u, \bar{p})) = u.$$

As with the classical Monge-Ampère equation, it is advantageous to restrict attention to the class of *G-convex* solutions u, that is, those functions that can be expressed in terms of some dual function $v : Y \mapsto \mathbb{R}$ and G via the generalized Legendre transform defined by

$$u(x) = \sup_{y\in Y} G(x, y, v(y)). \tag{1}$$

Then, the map $T_u(x)$ (under additional conditions) is characterized by

$$T_u(x) = y, \quad \text{where } y = \underset{y\in Y}{\operatorname{argmax}}\; G(x, y, v(y)). \tag{2}$$

The classical Monge-Ampère equation corresponds to the generating function $G(x, y, z) = -\langle x, y\rangle + z$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, in which case u is a convex function and $T_u(x) = \nabla u(x)$.

1.1 Applications of Generated Jacobian Equations

Let us elaborate further on the ways GJE arise in areas such as geometric optics, differential geometry, mathematical physics, and economics. We could start by pointing out that the wide applicability of GJE may already be inferred from the fact that GJE "contains" Optimal Transport as a "special case", however, this does not seem like a fair or illustrative approach. A more interesting measure of the merits for GJE is given by the many examples of problems in GJE are not treatable by the optimal transport framework.

Let us start with an important class of examples of GJE, namely the "near field" reflector problems in geometric optics, which fall outside the scope of OT[1]. In fact, at a previous BIRS workshop (Workshop # 12w5118), Trudinger posited the concept of GJE and proposed it as a framework that covers both optimal transport as well as near field reflector problems. The report for this workshop [] remarks that "a natural question is how to extend the results known up to now for optimal transport maps, to this more general setting, and it seems to give an interesting and challenging line of research".

One particular example is the point source near field reflector problem. In this problem there is a set of directions $X \subset \mathbb{S}^2$ through which light emanates from a point source at the origin O, and there is a region $Y \subset \Sigma$ in a smooth surface Σ. The goal is to illuminate Y by constructing a mirror to catch and reflect the light emitted by the source. Since the problem will be underdetermined otherwise, there are also two absolutely continuous mass distributions with densities $f(x)$ and $g(y)$ on X and Y respectively, with equal masses, f representing the initial distribution of light through X and g the desired output pattern on Y.

Then make the choice

$$G(x, y, z) = \frac{z^{-2} - \frac{1}{4}|y|^2}{z^{-1} - \frac{1}{2}(x, y)}. \tag{3}$$

Here, the radial graph of $G(\cdot, y, z)$ over $\mathbb{S}^2$ is an ellipsoid of revolution whose foci are at O and y, and whose eccentricity is determined from z. Then, a G-concave function u (defined as in (1), but with an inf instead

[1]"far field" problems, which are a limiting case of the near field regime, can be understood via OT

of a sup) represents a reflective surface given by envelope of ellipsoids of revolution with one foci at O and the other at points in Y. Assuming the reflective surface obeys Snell's law, a ray emanating from $\mathcal{O}$ in the direction x will bounce off the reflective surface and eventually hit Σ at a point $T_u(x) \in Y$, where $T_u(x)$ is as in (2) but with an argmin instead of argmax. Then in order to construct the desired mirror, one should take the radial graph of a G-convex solution $u : X \mapsto \mathbb{R}$ of the GJE

$$\det DT_u(x) = f(x)/g(DT_u(x)). \tag{4}$$

By considering other source and target domains X and Y, and different G corresponding to other basic reflective surfaces (e.g. paraboloids or cartesian ovals) one may treat many different near field geometric optic problems, such as parallel beam sources, or refraction problems.

In another direction, note all of the geometric optic problems above involve determining a surface whose normal vector behaves in such a way so that the ray tracing map sends one measure into another. This situation is very similar to that of prescribing the Gauss curvature of a convex surface (the Minkowski problem), which leads us to differential and convex geometry. A number of problems in convex analysis and convex geometry, generalizing the Minkowski problem, all lead naturally to Monge-Ampère type equations and generating functions. The Minkowski problem, which involves finding a closed convex surface with prescribed Gauss curvature, may, be posed as the analysis of a function of the form

$$\rho(x) = \inf_y G(x, y, z(y)), \quad G(x, y, z) := \frac{z}{x \cdot y}, \quad x, y \in \mathbb{S}^2,$$

for some strictly positive function $z(y)$ (and $G(x, y, z)$ being defined only for $x, y \in \mathbb{S}^2$ with $x \cdot y > 0$). There are also other related problems of great interest in the study of convex bodies, the L^p- and log-Minkowski problems. It is reasonable to expect that an analogue of this problem when $\mathbb{R}^3$ is replaced by a general Riemannian manifold will also lead to a GJE.

Alternatively, from the viewpoint of dynamics an important notion –currently limited to OT, with no parallel at the more general level of GJE– is that of **displacement convexity** and **Wasserstein barycenters**. Recall that if (X, d) is a metric space, OT imbues the space $\mathcal{P}_2(X)$ (probability measures on X with finite second moment) with a metric $d_{\mathrm{MK},2}$, defined via the Kantorovich problem

$$d_{\mathrm{MK},2}(\gamma_0, \gamma_1)^2 := \inf_{\pi \in \Pi(\gamma_0, \gamma_1)} \int_{X \times X} d(x, y)^2 \, d\pi(x, y),$$

where $\Pi(\gamma_0, \gamma_1)$ is the space of probability measures in $X \times X$ whose left and right marginals are γ_0 and γ_1, respectively. Then $d_{\mathrm{MK},2}$ makes $\mathcal{P}_2(X)$ a Polish and/or geodesic space as long as (X, d) is itself a Polish and/or geodesic space respectively, and thus in the latter case one can consider geodesics via this metric between two probability measures on X.

Displacement convexity is a property of functionals E on $\mathcal{P}_2(X)$, requiring the convexity of E along geodesics in this $d_{\mathrm{MK},2}$ metric. This notion of convexity for functionals is particularly useful in analyzing gradient flows for functionals on $\mathcal{P}_2(X)$ (such as relative entropy, as a famous example) and in particular for understanding the rate of convergence to equilibrium.

Lastly, let us discuss the emergence of generated Jacobian equations in economics, concretely, in relation to both matching problems and principal/agent problems. In either case, the set up involves compact metric spaces X and Y, and (in the greatest generality) a generating function $\phi : X \times Y \times \mathbb{R} \mapsto \mathbb{R}$, which is assumed to be continuous, strictly decreasing in its third argument, and such that $\phi(x, y, \mathbb{R}) = \mathbb{R}$ for all x, y. Moreover, ψ represents the inverse generating function. In this context, ϕ and ψ will represent the *utility functions* of certain parties.

In the *matching* context, $\phi(x, y, v)$ represents the utility that an agent x obtains when matched with agent y who is receiving a utility of v; the quantity $\psi(x, y, u)$ then corresponds to the utility agent y obtains when matched with agent x, who is receiving a utility of u. By contrast, in the *principal-agent* context $\phi(x, y, v)$ corresponds to the utility earned by an agent of type x who chooses decision y and makes a transfer of v to the principal, and $\psi(x, y, u)$ is the transfer which yields a utility of u to an agent of type x who chooses decision y. One is given distributions for agents x and agents y in the former case, and a utility function for the principal and a distribution for the agents in the latter one, and the goal is to find a *stable matching*: an admissible matching between parties where no pair has incentive to change their assigned pairing. Utility functions ϕ are called *quasilinear* if they are of the form $\phi(x, y, z) = b(x, y) + z$, for some $b(x, y)$, and previous work has been done connecting OT to this case of quasilinear utility. However, it can be more natural to assume a non-quasilinear utility, which now becomes part of the realm of general GJEs.

2 Recent Developments and Open Problems

It seems that generated Jacobian equations and the analysis of G-convex functions first appeared in the literature in two independent, vastly different contexts. The first is in the PDE literature motivated by near-field geometric optics problems, and the second in the economics literature in relation to principal-agent and matching problems in the non-quasilinear setting.

On the PDE side, in [34] Trudinger introduced GJE as a convenient framework covering a broad type of problems –motivated by geometric optics problems that are not covered by OT. Up until this point, *far-field* optics problems had been recognized as being covered by the OT framework (for example, [35]). The work of Trudinger in [34], besides introducing the GJE framework, studies existence of weak solutions subject to the second boundary value condition (the natural boundary condition when one prescribes the action of a volume deforming transformation), and provides (C^2 and higher) regularity results for smooth, positive densities under natural structural conditions on the generating function and domains. One of the conditions is a natural analogue of the Ma-Trudinger-Wang condition which plays a central role in regularity of OT ([28]). Since then, Guillen and Kitagawa have developed regularity theory ($C^{1,\alpha}$) for weak solutions of GJE with possibly discontinuous densities, while Jhaveri has obtained partial regularity results under minimal assumptions (see also discussion of presentations below). In terms of pure geometric optics, Gutierrez and Tournier analyzed the regularity for the near field parallel reflector/refractor problem [19] and also studied problems for media with different refractive indices [18]. Gutierrez and Sabra also studied models for scattering on free form lenses [17] and aspherical lenses [16]. There is also work of Karakhanyan and Wang [20] regarding the near field point source problem, which obtains regularity of reflectors for smooth densities under natural structural assumptions, but furthermore, provides an example of singular behavior of a nature not exhibited in OT. This brief and incomplete lists illustrates just some of the recent activity from the PDE side concerning GJEs, and the many geometric optic problems that GJE encompasses.

Independently, from the economics community, Nöldeke and Samuelson [32] initiated a systematic study of generating functions and G-convex analysis in the context of the "implementation duality" in matching and principal-agent problems. Previous works [5] and [12] analyzed the quasilinear case, which falls within the realm of OT, but the work of Nöldeke and Samuelson deals specifically with utility functions which are not quasilinear. Another interesting development is in relation to the *second welfare theorem*, which in the quasilinear case essentially indicates that a matching in equilibrium coincides with the optimality conditions arising in OT. In the non-quasilinear case, non-transferable utilities may make it impossible to characterize equilibrium matchings as optimal, which has led Galichon to propose "equilibrium" as a replacement for "optimal" in matching problems. This in turn leads to a question that arises naturally and has connections with geometric optics as well: what is the analogue for GJEs of Kantorovich's relaxation of OT? This is an unresolved question, in fact, it is not even clear what a corresponding notion of a transport *plan* should be within GJE. This issue not withstanding, work by Liu [26] shows there is a variational characterization behind certain reflector antenna problems, and shows this question can form a unifying thread between economics and geometric optics within GJE.

Finally, we mention the great deal of activity relating to numerical schemes for GJEs. There is work in the semi-discrete setting by Merigot, Meyron, and Thibert [31] as well as by Kitagawa, Merigot, and Thibert [25] for optimal transport. Monotone approximation schemes for viscosity solutions of the Monge-Ampère equation have been obtained recently by Froese and Oberman [13], which are notable for their handling of the second boundary value problem. Developing respective numerical schemes for various geometric optic problems is a worthwhile research direction that is still in its early stages.

3 Presentation Highlights

This workshop brought together mathematicians (notably, experts in differential geometry, PDE, convex analysis), engineers and applied mathematicians (numerical analysis, optimization, mathematical physics), and economists (stable matching theory, principal/agent problems). Our chief goals were to compare each field's methods, research directions, and conjectures, and to lay the ground for future interdisciplinary work.

3.1 General Theory of GJEs

Nestor Guillen began with an overview of generated Jacobian equations, presenting the basic setup and terminology as first introduced by Trudinger in [34]. This included a brief overview of the applications to geometric optics and economics, a discussion of his recent work in regularity of weak solutions with Kitagawa [15], and key differences with the classical Monge-Ampère and optimal transport theory (as shown by Karakhanyan and Wang [20]).

Yash Jhaveri discussed a partial regularity result for GJE, showing that for generating functions that satisfy structural conditions and continuous, bounded data, there are two closed sets of measure zero, outside of which the associated transport mapping is a $C^{0,\alpha}$ homeomorphism. One significance of this result is that it does not require an MTW like condition that is usually necessary for full regularity results. The result first appeared in the context of the real Monge-Ampère equation by Figalli and Kim [11] and OT by De Philippis and Figalli [10].

Brendan Pass presented recent work with McCann on OT problems between spaces of unequal dimension. They have shown that when the dimension of the source domain is larger than that of the target domain, the problem can be described by a non-local analogue of the Monge-Ampère equation that arises for transport between spaces of equal dimensions. Such problems are of great interest in economic matching problems, as the spaces to be matched (buyers and sellers, employers and employees, etc.) naturally depend on different numbers of parameters. In another recent work with Chiappori and McCann [6], Pass has exhibited some conditions under which one can obtain uniqueness and regularity of transport maps when the target space is one dimensional.

3.2 Numerical Analysis

Boris Thibert presented two numerical schemes for the semi-discrete optimal transport problem based on damped Newton methods. The first is applicable to transportation of an absolutely continuous measure to a discrete measure subject to a cost function satisfying the MTW condition, under only mild connectivity assumptions on the source of the source measure (joint with Kitagawa and Mérigot [25]). This scheme relies on the geometric implications of convexity coming from the MTW condition, which was first shown by Loeper ([27]). The second considers the case where the source measure is supported on a simplex soup and the target is discrete (joint with Mérigot and Meyron [31]). In this case the convex structure of the MTW case is lost, hence the scheme requires the points in the support of the discrete target satisfy a natural non-degeneracy type condition. Both results include rigorous proofs of convergence rates.

Farhan Abedin discussed an iterative method for computing solutions to GJE also in the semi-discrete case. The method is in the vein of Caffarelli, Kochengin, and Oliker's in the far-field reflector problem [3] and Kitagawa's for strongly MTW costs [24], but Abedin's method is novel in that it does not require any MTW like conditions on the generating function to obtain termination after a finite number of steps (which is new even in the OT case) and thus applies to an extremely wide variety of generating functions.

Brittany Froese introduced a general framework to construct monotone approximation schemes for Monge-Ampère type equations given a point cloud satisfying certain structural conditions. With this framework and a careful resolution of boundary geometry, given a degenerate elliptic equation with a comparison principle, Froese is able to construct approximation schemes which converge to viscosity solutions of the PDE [14]. Numerical results were also presented for examples including the prescribed Gauss curvature problem with Dirichlet boundary data, the quadratic optimal transport problem, and problems arising from seismic imaging and beam shaping.

3.3 GJE and OT in economics

Larry Samuelson's talk dealt with generalized notions of convexity and how they are a means to understand the "implementability" of assignment maps in natural problems in economics, again in instances where the underlying utilities may not be quasilinear. A natural concept is that of a *profile* $v : Y \mapsto \mathbb{R}$ *implementing* a pair (u, y), where $u : X \mapsto \mathbb{R}$ (another profile) and $y : X \mapsto X$ (an *assignment*), which means that

$$u(x) = \max_{y \in Y} \phi(x, y, v(y)), \quad y(x) \in \text{argmin}_{y \in Y} \phi(x, y, v(y)).$$

In the matching model, u and v are profiles of utilities for the buyers and sellers. In the principal-agent model, u is a rent function for the agent, giving a utility $u(x)$ for agent x, and v is a tariff function, giving the tariff $v(y)$ at which any agent can execute contract y. One result presented by Samuelson was that the set of pairwise, stable, full outcomes satisfying a given initial condition (y_1, v_1) is nonempty and closed in the right topology. Additionally, he mentioned that implementation maps are examples of a (antitone) **Galois connection**: a pair of order reversing mappings between posets of a set.

Alfred Galichon introduced the framework of *equilibrium transportation*, which encompasses problems arising naturally in economics which often fall outside the scope of OT. As mentioned in the previous section, the second welfare theorem gives the equivalence of stability and optimality in matching problems involving quasilinear utilities, but this may not be the case in the non-quasilinear case. However, Galichon posits that it does make sense to talk about equilibrium in matchings in the non-quasilinear case, and perhaps this is the correct notion to pursue rather than optimality. Galichon illustrated this situation by introducing a variation on the matching problem involving *taxation*. In this setting, workers x are paired with firms y and receive a wage $w(x, y)$ while the worker obtains a utility $\alpha(x, y)$ (this may represent aspects different from pay, such as job satisfaction). The firm obtains a utility $\gamma(x, y)$, hence their total profit is $\gamma(x, y) - w(x, y)$. However, there is a tax levied on the worker's pay, thus the actual benefit a worker gains is $\alpha(x, y) + N(w(x, y))$ for some function N. The goals is then to match a group of workers with firms in a stable manner, in the case of no taxation ($N(w) = w$) we recover OT and stability is equivalent to optimality. In the case of taxation, Galichon introduces the notion of *feasible sets* defined by

$$\mathcal{F}_{xy} = \{(u, v) \in \mathbb{R}^2 : u - \alpha(x, y) \leq N(\gamma(x, y) - v)\},$$

and a triple (P, u, v) where $P \in \mathcal{P}(X \times Y)$, u, $v : X, Y \to \mathbb{R}$ is said to be an equilibrium when

$$(u(x), v(y)) \notin \mathcal{F}^{\circ}_{xy} \text{ and } (x, y) \in \text{spt}(\pi) \Rightarrow (u(x), v(y)) \in \mathcal{F}_{xy}. \tag{5}$$

Finding (π, u, v) such that (5) holds is known as a *Nonlinear Complimentary Problem* (NCP). In the continuum case, if the matching between x and y is given by a smooth map $x \to y(x)$, then $y(x)$ is the G-gradient map of a G-convex function u which (using the mass balance) solves a corresponding GJE. The map $y(x)$ and scalar $u(x)$ are examples of an implementation map, as discussed by Nöldeke and Samuelson [32].

Shuangjian Zhang discussed a principal-agent problem modeling a pricing problem for a monopolist involving non-quasilinear utility functions, and presented necessary and sufficient conditions for the principal's problem to be a linear program (joint with McCann, [30]). Here the principal-agent problem is where a principal and agents attempt to maximize their utilities which arise through the execution of certain contracts, but often with some amount of information asymmetry. In the context of Zhang and McCann's work the principal is a monopolist manufacturer and agents are potential buyers, the different contracts to be executed are the purchase of different models of whatever product the principal manufactures. The monopolist then has a (possibly non-quasilinear) profit function $\pi(x, y, v)$, which is the profit generated by agent x buying the product y at price v. Then the monopolist knows the distribution $d\mu$ of different types of buyers in the market and the utility that a given buyer type derives from purchasing a given model of product, and seeks to set the price $v(y)$ for each model y in a manner which maximizes their profit, i.e. they wish to maximize the functional

$$\int_X \pi(x, y(x), v(y(x)))\, d\mu(x),$$

where here $y(x)$ is the choice of model that a buyer x will select based on their own utility and a given pricing scheme. Zhang has shown that under a convexity condition on the agents utility function along certain curves, the principal's problem is a linear program. This convexity condition follows from a strengthening of the analogue of MTW condition for GJEs, which corresponds to the condition of non-negative cross curvature used in previous work of Figalli, Kim, and McCann ([12]) for the case of quasilinear utilities.

Guillaume Carlier presented a model for equilibrium prices with constraints stemming from work with Ekeland and Galichon. In the model, one has a set Z representing feasible goods, and which is assumed to be a compact metric space. The goal is to determine a price system that clears the market, that is, a price function $p : Z \mapsto \mathbb{R}$ which leads to equilibrium.

One is given a set of producers $y \in Y$ (assumed also a compact metric space), a cost function $c : Y \times Z \mapsto \mathbb{R}$ (what it costs to y to produce z), and a distribution of producers corresponding to a probability distribution ν over Y. Then, given $p(z)$, it's c transform is $p^c(y) = \max_z\{c(y,z) - p(z)\}$ and it represents what the maximum cost to producer y when selling good z in this price system. Consumers are represented by a compact metric space X with a probability distribution $\mu \in \mathcal{P}(X)$.

So far, this is standard, the novelty arises in that consumers are modeled to have a budget: each consumer x has a revenue $\phi(x)$, $\phi \in C(X)$, as well as a utility function given by $U : X \times Z \mapsto \mathbb{R}$, such that $U \in C(X \times Z)$. Then, given a price function $p(z)$, we define

$$U_p(x) = \max\{U(x,z) : z \text{ s.t. } p(z) \leq \phi(x)\}$$

Then, the problem is to determine a price controls p and matching plans for consumers/goods $\gamma \in \mathcal{P}(X \times Z)$ and producers/goods $\sigma \in \mathcal{P}(Y \times Z)$ satisfying the following. The prize function is such that $p \in C(Z, \mathbb{R})$ and $\min_Z p(z) = \min_X \phi$, while $\pi_x \# \gamma = \mu$, $\pi_y \# \sigma = \nu$, and $\pi_z \# \gamma = \pi_z \# \sigma$. Then, the equilibrium corresponds to these objects satisfying

$$p^c(y) + p(z) = c(y,z)\ \sigma\text{-a.e.}, \quad \text{while } p(z) \leq \phi(x) \text{ and } U(x,z) = U_\rho(x)\ \gamma\text{-a.e.}$$

This a case of nontransferable utility and OT methods do not apply. An existence result is obtained for equilibria in the above sense.

Beatrice Acciaio provided an introduction to the theory of causal transport and some of its applications, based on joint work with Backhoff and Zalashko [1]. In probabilistic language, classical OT corresponds to minimizing the expectation of the cost function:

$$\inf\{\mathbb{E}_\pi[c(x,y)]\ : P \in \Pi(\mu,\nu)\}$$

where as usual, $\Pi(\mu,\nu)$ denotes the space of probability measures over $X \times Y$ with marginals μ and ν. Acciaio introduced the variant of *causal transport*. Here one takes filtrations $\mathcal{F}^X$ and $\mathcal{F}^Y$ over X and Y respectively, then $\pi \in \Pi(\mu,\nu)$ is said to be a *causal plan* if for all t and $D \in \mathcal{F}^Y_t$ the map $x \mapsto P^x(D)$ is measurable w.r.t. to $\mathcal{F}^X_t$. Then, the causal transport problem is the minimization problem

$$\inf\{\mathbb{E}_\pi[c(x,y)]\ : P \in \Pi(\mu,\nu) \text{ such that } P \text{ is causal}\},$$

corresponding to a constrained version of the classical OT problem where time matters, hence the involvement of causality. Attainability for the causal transport problem, as well as the expected duality relations are satisfied. Acciaio also discussed an application of causal transport, to the following question of importance in stochastic integration: given a space of events Ω imbued with two filtrations $\mathcal{F}$ and $\mathcal{G}$, and a probability measure $\mathbb{P}$, then given X a semimartingale in $(\Omega, \mathcal{F}, \mathbb{P})$ when can we say that X is also a semimartingale in $(\Omega, \mathcal{G}, \mathbb{P})$? With an appropriate choice of cost function, causal transport provides an optimal transport characterization for those processes B which are a Brownian motion with respect to a given filtration and remain a semimartingale with respect to an enlarged filtration.

3.4 Dynamics

Bernhard Schmitzer's presentation was concerned with the problem of "unbalanced" optimal transport and various approaches. This problem, considered in joint work with Benedikt Wirth [33], is relevant to situations where the two measures under consideration mays not have the same total mass –e.g. a partial transport problem or an inference problem where one needs to estimate the similarity between two distributions that may have different total mass. Such problems can arise for example in image interpolation between images that are not normalized (such as medical images). Schmitzer's talk considered both *static* as well as *dynamic* formulations (much in the spirit of work of Benamou and Brenier). Such considerations are important as frequently unbalanced transport is initially defined via a dynamic formulation, such as in the case of the Wasserstein-Fisher-Rao metric, and a major question is if there is a corresponding static formulation. Schmitzer and Wirth's work deals with an interpolation involving the OT metric derived from the distance cost, and not from the distance squared as in the case of Wasserstein-Fisher-Rao.

Codina Cotar presented recent developments in the connections between multimarginal optimal transport and density functional theory. Density functional theory or DFT, is a simplified version of quantum mechanics in which an N-particle system (represented by a probability distribution on $\mathbb{R}^{3N}$) is described by the behavior of its one-particle marginals on $\mathbb{R}^3$, and models electron interactions of molecules as described by the Schrödinger equation.

In recent joint work with Friesecke and Klüppelberg [7], Cotar has investigated an approximate DFT ground state energy functional where the electron-electron interaction in the Hohenberg-Kohn functional is replaced with one coming from optimal transport. This transport is a *multimarginal* transport problem with a Coulomb cost function, of the form $c(x_1, \ldots, x_N) := \sum_{i \neq j} |x_i - x_j|^{-1}$. In the above joint work, Cotar has shown that this optimal transport problem has a unique solution, the minimal energy in the DFT approximation is always a lower bound for the true quantum mechanical ground state energy, and in the two particle case this optimal transport energy gives the semiclassical limit of the Hohenberg-Kohn functional. Finally, it was announced that there is an ongoing joint work with Petrache in which they are analyzing higher order corrections to this semiclassical limit in the general N-particle case.

Katy Craig presented recent results concerning existence, uniqueness, and stability of solutions to PDE involving both degenerate diffusion and aggregation. Specifically, the focus was on equations of the form

$$\partial_t \rho = div((\nabla K * \rho)\rho) + \Delta(\rho^m)$$

which arise in many contexts such as biological chemotaxis, swarming, and modeling granular media. Such equations are known to correspond, at least formally, to gradient flows of certain energies under the Wasserstein metric. Questions such as existence and uniqueness are known for cases where the kernel K is λ-convex for some $\lambda \in \mathbb{R}$, but there are many interesting examples (such as those raised above) that do not fall in this framework. In [8], Craig has shown existence, uniqueness, and quantitative stability estimates (double-exponential bounds in time) for gradient flows of an energy functional that is ω-convex, for some modulus of convexity ω satisfying the Osgood criterion. In further joint work with I. Kim and Y. Yao [9], Craig uses this framework to analyze the Keller-Segel equation with a hard height constraint (which can be viewed as a limiting case of the above diffusion-aggregation equation as $m \to \infty$).

Micah Warren discussed some new ideas related to a mean curvature flow of the pseudo-Riemannian structure given by optimal transport. In 2010, Kim and McCann [21] introduced a pseudo-Riemannian metric defined using the cost function in an optimal transport problem, which in particular captures the MTW condition as positivity of certain sectional curvatures under this metric. In an interesting development, Kim, McCann, and Warren [22] showed that minimizers in the optimal transport problem give rise to codimension n, volume *maximizing* submanifolds under the pseudo-Riemannian metric of Kim and McCann (after a conformal change). Warren suggests then that a natural line of investigation is to consider the gradient flow of the volume functional, which leads to the pseudo-Riemannian mean curvature flow. It is noted that this could be a very promising direction, as generally mean curvature flows tend to be better behaved in the pseudo-Riemannian framework compared to the positive definite case, as in for example, a result of Li and Salavessa regarding the second fundamental form of space-like manifolds under mean curvature flow in pseudo-Riemannian spaces.

3.5 Geometric optics

Cristian Gutiérrez presented recent results on free-form lens design. Taking the point source refractor problem as an example, traditional methods encase the light source in some medium, then cut a ball out centered at the light source to create a lens. However, in practice such a model results in very thick, large lenses, which are heavy and difficult to manufacture. Thus a new goal is to construct a lens sandwiched between two optically active surfaces to focus a light source into some desired energy distribution, which can potentially be made thinner. Gutiérrez discussed recent joint work with Sabra [16, 17] in which they show that given one optically active surface, they can solve a system of PDE in order to find a corresponding second surface which will combine with the first to create such a desired lens, in the far-field regime. They give conditions under which the representation of this second surface is physically realizable, along with estimates that can be given for the thickness of the resulting lens. It is noted than an important question that remains is to find an appropriate first surface so that the second surface may be relatively simple to manufacture.

Ahmad Sabra presented results on an alternative framework to construct optical surfaces in the near-field regime when the usual convex model may face physical obstructions. Using the near-field parallel, point source reflector problem as an example, Sabra discussed how the usual model of constructing a reflecting surface as intersections of ellipsoids may encounter obstructions such as self-blocking of reflected rays by the surface itself. In recent work with Gutiérrez, he proposes to instead take unions of ellipsoids which will result in reflectors that are no longer convex. Sabra is able to prove existence of such reflectors when the target energy distribution is discrete, and then pass to a limit in order to recreate absolutely continuous densities. It should be noted that regularity for this model is still unexplored and falls outside of the usual elliptic framework, in fact it would give rise to a GJE of mixed hyperbolic - elliptic type.

3.6 Geometry

Aram Karakhanyan talked about generalizations of Blashke's *rolling ball theorem*. The original theorem deals with the following geometric situation (which explains the name of the theorem), we are given M and M', two smooth and strongly convex surfaces whose second fundamental forms II_M and $\mathrm{II}_{M'}$ are such that $\mathrm{II}_M(x) \geq \mathrm{II}_{M'}(x')$ whenever $x \in M$ and $x \in M'$ are such that the outer unit normal vectors satisfy $n_M(x) = n_{M'}(x')$. Blashke's theorem says that if M and M' are internally tangent at one point then M is contained in the convex region bounded by M'. In the OT community, this statement is reminiscent of the local-to-global property for costs satisfying the MTW condition.

Karakhanyan presented a generalization of this result where instead, one considers M being strongly c-convex and M' a c-hyperplane with respect to a cost function c satisfying certain natural structural assumptions. He also demonstrated one important application of this result, to parallel reflector and refractor problems.

Young-Heon Kim presented a new attempt at defining a canonical notion of the barycenter of a measure on a metric measure space, via a process of *Wasserstein regularization*. In joint work with Pass [23], if $\mu \in \mathcal{P}(M)$ for some metric measure space (M, d, m), and $\epsilon > 0$, then they consider the problem of minimizing the functional

$$\nu \mapsto \int_X \int_X d^2(x,y) d\mu(x) d\nu(y) + \varepsilon W_2^2(\nu, m)$$

over $\nu \in \mathcal{P}(M)$, where here W_2 is the 2-Wasserstein distance between probability measures on M. For each $\epsilon > 0$, there exists a unique minimizer μ_ϵ, and moreover the μ_ϵ converge weakly to some limiting measure $B(\mu)$ as $\epsilon \searrow 0$, which is supported in the set $b(\mu)$, the set of barycentric points of μ that minimize the quantity $\int_M d^2(x, \cdot)\mu(dx)$. Moreover, $B(\mu)$ can uniquely be characterized as the minimizer of $W_2(\cdot, \mu)$ over probability measures supported on $b(\mu)$. Additionally, Kim and Pass have analyzed some of the dynamics that the mapping $\mu \mapsto B(\mu)$ induces over the space $\mathcal{P}(M)$ of probability measures, including fixed points and periodic orbits.

Yi Wang presented recent developments on a nonlinear Sobolev trace inequality involving the k-Hessian energy in place of the usual Dirichlet energy. It is a classical result that if u is a function on Ω with $u = f$ on $\partial\Omega$ and u_f is the harmonic extension of f to Ω, then

$$\int -u\Delta u \, dx + \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u \partial_\nu u \, \sigma(dx) \geq \frac{1}{|\partial\Omega|} u_f \partial_\nu(u_f) \, \sigma(dx)$$

where ν is the outward unit normal and σ the surface measure on $\partial\Omega$. One question that arises is if the first term above (which corresponds to the 1-Hessian energy) can be replaced by the k-Hessian energy for $k > 1$, and in this case what the appropriate choice of boundary operator to replace ∂_ν may be. In joint work with Case [4], Wang has shown that such a boundary operator exists, along with a corresponding "polarized" version of the inequality, yielding a sharp Sobolev trace inequality for k-admissible functions.

Robin Neumayer discussed a one parameter family of variational problems on the upper half space in $\mathbb{R}^n$, where certain values of the parameter correspond to optimizers of the classical L^p Sobolev inequality with zero boundary condition, and the L^p Sobolev trace inequality. In joint work with Maggi [29], in the case of $n \geq 2$ and $p > 1$, Neumayer succeeded in characterizing minimizers for all positive values of the parameter, along with a detailed analysis of the properties of the optimal value as a function of the parameter. Additionally when $p = 2$ and $n \geq 3$, the variational problem can be interpreted as minimizing certain curvature quantities over a class of conformal changes of the Euclidean metric, and the aforementioned characterizations of minimizers prove that in for various ranges of the parameter one recovers exactly the spherical, flat, and hyperbolic geometries.

Deane Yang gave an overview of the current state of the L^p Brunn-Minkowski theory. The L^p Minkowski problem is, given a Borel measure μ on $\mathbb{S}^{n-1}$, to find a convex body K such that the L^p surface area measure of the body is μ. Here, the L_p-surface area of a convex body K with outer normal ν_K is a Borel measured over $\mathbb{S}^{n-1}$ given by

$$S_p(K,\omega) := \int_{x\in{\nu_K}^{-1}(\omega)} (x\cdot\nu_K(x))^{1-p}\, d\mathcal{H}^{n-1}(x), \;\; \forall\, \omega \in \mathcal{B}(\mathbb{S}^{n-1}).$$

The case $p = 1$ corresponds to the classical Minkowski problem of prescribed Gauss curvature, and for $p \geq 1$, the problem can be attacked with a variational approach. Yang presented recent results on a particularly interesting case which is when $p \to 0$, which is the logarithmic Minkowski problem of prescribing the cone volume measure of a body. In joint work with Böröczky, Lutwak, and Zhang [2], Yang has shown that under something called the subspace concentration inequality, the log Minkowski problem has a solution. Finally, Yang closed with a question, asking if the case corresponding to $p = -n$ can be recast as an optimal transport or generated Jacobian equation, which could provide a new tool for this difficult and important case.

4 Reception of the workshop and participant's feedback

Given the reception of the workshop, it is safe to say that the goal of bringing several different communities of researchers together was achieved successfully. In particular, several new collaborations were initiated at the conference. Katy Craig reports that motivated by the discussions during the conference, she has started a project with Micah Warren where they will be exploring connections between the Jordan-Kinderlehrer-Otto scheme from gradient flows and game theory. Galichon and Carlier indicated that as a result of the workshop they have started on a new project involving a hedonic model with tight budget constraints. Abedin expressed that as a result of the workshop he is considering embarking on a project involving regularity for GJE equations under a strong MTW type condition.

There was also a very successful open problem session which was originally planned to last for one hour, but ended up going for twice that. Problems were proposed by Alfred Galichon, Jun Kitagawa, Aram Karakhanyan, Ahmad Sabra, and Alessio Figalli.

The participants' feedback was extremely positive and enthusiastic, with many highlighting the interdisciplinary nature of the workshop, here we present quotes from four different participants:

I learned about the fascinating parallel between problems in reflector design in geometric optics and problems on taxation in matching markets.

The connection between generated Jacobian equations and economics seems very intriguing and hopefully will spark some ideas for future research.

My work is focused on economics, and before this conference, I was unaware of both the vast theoretical literature in economics as well as applications to optics and other areas. The conference was an invaluable point of entry into these areas, accomplishing in a few days what would have taken me months or years to do on my own. I will make good use of what I've learned in my continuing research.

The introductory lecture was very useful, helping me put many of the subsequent talks into context. The open problem session was also great.

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