The Kneser–Poulsen Conjecture for Uniform Contractions

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Notation, terminology

$\mathbf{p} = (p_1, \ldots, p_N) \in \mathbb{E}^{d \times N}$: a configuration (ie. a set, or a sequence) of N points in Euclidean d-space $\mathbb{E}^d$.

$\mathbf{q} \in \mathbb{E}^{d \times N}$ is a contraction of $\mathbf{p} \in \mathbb{E}^{d \times N}$, if $|q_i - q_j| \leq |p_i - p_j|$ for all $1 \leq i < j \leq N$.

$\mathbf{B}[\mathbf{p}] = \bigcap_{i \in N} \mathbf{B}[p_i, 1]$.

Kneser–Poulsen Conjecture ~'54

If $\mathbf{q} = (q_1, \ldots, q_N)$ is a contraction of $\mathbf{p} = (p_1, \ldots, p_N)$ in $\mathbb{E}^d$, then

$$\mathrm{V}_d\left(\bigcap_{i=1}^{N} \mathbf{B}[p_i]\right) \leq \mathrm{V}_d\left(\bigcap_{i=1}^{N} \mathbf{B}[q_i]\right). \tag{KP}$$

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Alexander's Conjecture '85

If $\mathbf{q}$ is a contraction of $\mathbf{p}$ in $\mathbb{E}^2$, then

$$\mathrm{perim}(\mathbf{B}[\mathbf{p}]) \leq \mathrm{perim}(\mathbf{B}[\mathbf{q}]) . \tag{A}$$

Habicht and Kneser: for unions, the reversal (A) is FALSE.

Uniform contraction

$\mathbf{q} \in \mathbb{E}^{d\times N}$ is a uniform contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$ with separating value λ, if

$$|q_i - q_j| \leq \lambda \leq |p_i - p_j| \text{ for all } 1 \leq i < j \leq N. \qquad \text{(UC)}$$

Motivation

P. Pivovarov's idea to disprove (KP): sample **p** and **q** randomly. Show that with $\neq 0$ probability, (KP) is false, while (UC) holds.
[Paouris-Pivovarov, *Random ball-polyhedra and inequalities for intrinsic volumes*, Monatshefte, 2016].

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Main result

$k \in [d]$. Let $\mathbf{q} \in \mathbb{E}^{d\times N}$ be a uniform contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$ with any separating value $\lambda \in (0, 2]$. If $N \geq \left(1+\sqrt{2}\right)^d$ then

$$\mathrm{V}_k\left(\mathbf{B}[\mathbf{p}]\right) \leq \mathrm{V}_k\left(\mathbf{B}[\mathbf{q}]\right). \qquad (1)$$

A bit stronger:

Theorem

$k \in [d]$. Let $\mathbf{q} \in \mathbb{E}^{d\times N}$ be a uniform contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$ with any separating value $\lambda \in (0, 2]$. If

(a) $N \geq \left(1 + \frac{2}{\lambda}\right)^d$,

or

(b) $\lambda \leq \sqrt{2}$ *and* $N \geq \left(1 + \sqrt{\frac{2d}{d+1}}\right)^d$,

then (1) *holds.*

Unions

Theorem

Let $\mathbf{q} \in \mathbb{E}^{d \times N}$ *be a uniform contraction of* $\mathbf{p} \in \mathbb{E}^{d \times N}$ *with some separating value* $\lambda \in (0, 2]$*. If* $N \geq \left(1 + 2d^3\right)^d$ *then*

$$\mathrm{V}_d\left(\bigcup_{i=1}^{N} \mathbf{B}[p_i]\right) \geq \mathrm{V}_d\left(\bigcup_{i=1}^{N} \mathbf{B}[q_i]\right).$$

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Proof

Heavy lifting done by Rogers and Bezdek–Lángi on soft ball packings.

Proof of the Main Result – Easy estimates

$$f_k(d, N, \lambda) := \min\left\{\mathrm{V}_k\left(\mathbf{B}[\mathbf{q}]\right) \;:\; \mathbf{q} \in \mathbb{E}^{d\times N}, |q_i - q_j| \le \lambda \;\; \forall i, j \in [N], i \ne j\right\},$$

$$g_k(d, N, \lambda) := \max\left\{\mathrm{V}_k\left(\mathbf{B}[\mathbf{p}]\right) \;:\; \mathbf{p} \in \mathbb{E}^{d\times N}, |p_i - p_j| \ge \lambda \; \forall i, j \in [N], i \ne j\right\}.$$

Goal: $f_k \ge g_k$.

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Jung's bound on f_k

Let $d, N \in \mathbb{Z}^+, k \in [d]$ and $\lambda \in (0, \sqrt{2}]$. Then

$$f_k(d, N, \lambda) \geq \left(1 - \sqrt{\frac{2d}{d+1}}\frac{\lambda}{2}\right)^k \mathrm{V}_k\left(\mathbf{B}[o]\right).$$

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Proof: $\mathbf{q}$ is contained in a ball of radius $\sqrt{\frac{2d}{d+1}}\frac{\lambda}{2}$. Thus, $\mathbf{B}[\mathbf{q}]$ contains a ball of radius ... □

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Packing bound on g_k

Let $d, N \in \mathbb{Z}^+, k \in [d]$ and $\lambda > 0$.

$$\text{If } N\left(\frac{\lambda}{2}\right)^d \geq \left(1 + \frac{\lambda}{2}\right)^d, \text{ then } g_k(d, N, \lambda) = 0.$$

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Proof: $\{\mathbf{B}[p_i, \lambda/2]\}$ is a packing. Thus, taking volume yields that the circumradius of the set $\{p_i\}$ is at least one. Hence, $\mathbf{B}[\mathbf{p}]$ is a singleton or empty. □

An additive Blaschke–Santalo inequality

$X \subset \mathbb{E}^d, \mathrm{cr}(X) \le \rho$. The ρ-spindle convex hull of X is $\mathrm{conv}_\rho(X) := \mathbf{B}[\mathbf{B}[X,\rho],\rho]$. Easily, $\mathbf{B}[X,\rho] = \mathbf{B}[\mathrm{conv}_\rho(X),\rho]$.

Fodor, Kurusa, Vígh: A Blaschke–Santalo-type inequality for the volume of spindle convex sets, [FKV, *Inequalities for hyperconvex sets*, Adv. Geom, '16].

A variation: an additive Blaschke–Santalo-type inequality for spindle-convex sets for intrinsic volumes.

Additive Blaschke–Santalo inequality

$Y \subset \mathbb{E}^d$ a ρ-spindle convex set, $k \in [d]$. Then

$$\mathrm{V}_k(Y)^{1/k} + \mathrm{V}_k(\mathbf{B}[Y,\rho])^{1/k} \le \rho \mathrm{V}_k(\mathbf{B}[o])^{1/k}. \qquad \text{(Bl.}\neq\text{Sa.)}$$

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Proof:

Proposition [folklore]

$Y \subset \mathbb{E}^d$ a ρ-spindle convex set. Then

$$Y - \mathbf{B}[Y,\rho] = \mathbf{B}[o,\rho].$$

Combine with the Brunn–Minkowski theorem for intrinsic volumes. □

Proof of the Proposition

$Y \subset \mathbb{E}^d$ a ρ-spindle convex set. Then

$$Y - \mathbf{B}[Y, \rho] = \mathbf{B}[o, \rho].$$

Y spindle-convex, thus, Y slides freely in $\mathbf{B}[o, \rho]$.
Thus, Y is a summand of $\mathbf{B}[o, \rho]$ and so,

$$Y + (\mathbf{B}[o, \rho] \sim Y) = \mathbf{B}[o, \rho],$$

where $\sim$ is the Minkowski difference: $\mathbf{B}[o, \rho] \sim Y := \cap_{y \in Y}(\mathbf{B}[o, \rho] - y)$.
On the other hand, $\cap_{y \in Y}(\mathbf{B}[o, \rho] - y) = -\mathbf{B}[Y, \rho]$. □

A non-trivial bound on g

$$g_k(d, N, \lambda) := \max \left\{ \mathrm{V}_k\left(\mathbf{B}[\mathbf{p}]\right) \ : \ \mathbf{p} \in \mathbb{E}^{d \times N}, |p_i - p_j| \geq \lambda \ \forall i, j \in [N], i \neq j \right\}.$$

$$g_k(d, N, \lambda) \leq \max \left\{ 0, \left(1 - \left(N^{1/d} - 1\right) \frac{\lambda}{2}\right)^k \mathrm{V}_k\left(\mathbf{B}[o]\right) \right\}.$$

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A simple fact: $\mathbf{B}[\mathbf{p}] = \mathbf{B} \left[\bigcup_{i=1}^{N} \mathbf{B}[p_i, \mu], 1 + \mu \right]$.

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$$\left[\left(1 + \frac{\lambda}{2}\right) \mathrm{V}_k\left(\mathbf{B}[o]\right)^{1/k} - \mathrm{V}_k\left(\mathrm{conv}_{1+\lambda/2}\left(\bigsqcup_{i=1}^{N} \mathbf{B}\left[p_i, \frac{\lambda}{2}\right]\right)\right)^{1/k}\right]^k \leq$$

Goal: $g_k(d, N, \lambda) \le \left(1 - \left(N^{1/d} - 1\right)\frac{\lambda}{2}\right)^k \mathrm{V}_k\left(\mathbf{B}[o]\right).$

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In the last step, we used:

$$\mathrm{V}_d\left(\mathrm{conv}_{1+\lambda/2}\left(\bigsqcup_{i=1}^{N} \mathbf{B}\left[p_i, \frac{\lambda}{2}\right]\right)\right) \geq \mathrm{V}_d\left((N^{1/d}\lambda/2)\mathbf{B}[o]\right), \qquad \text{(VOL)}$$

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and a general isoperimetric inequality: *among all convex bodies of a given volume, the ball has the smallest V_k.*

Thus, we can replace V_d by V_k in (VOL). □

Completing the proof of the Main Result

Combine the bounds on f_k and g_k. $\square$

Strong contractions

Unconditional body: symmetric about each of the d coordinate hyperplanes.

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Theorem

$K_1, \ldots, K_N$ unconditional convex bodies in $\mathbb{E}^d$. $\mathbf{q} \in \mathbb{E}^{d\times N}$ a strong contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$. Then

$$\mathrm{V}_d\left(\bigcup_{i=1}^{N}(p_i + K_i)\right) \geq \mathrm{V}_d\left(\bigcup_{i=1}^{N}(q_i + K_i)\right),$$

and

$$\mathrm{V}_d\left(\bigcap_{i=1}^{N}(p_i + K_i)\right) \leq \mathrm{V}_d\left(\bigcap_{i=1}^{N}(q_i + K_i)\right).$$

Picture time

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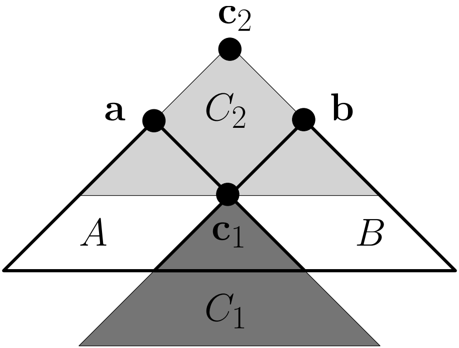

Figure :
1st family: A, B, C_1; 2nd family: A, B, C_2. Translation vectors: $b = -a$, $c_2 = -c_1$.

Both configurations of the three translation vectors are a strong contraction of the other configuration.

Does it work for the perimeter?

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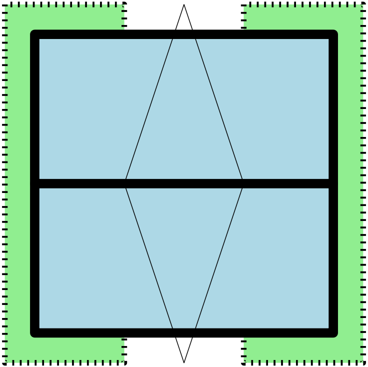

Figure :
1st family: The two green rectangles, the diamond, and the two blue rectangles.
2nd family: The two green rectangles, the diamond, and ONE blue rectangle (counted twice).

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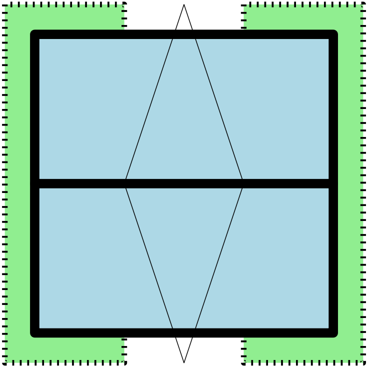

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Thank you!