Representing graphs by sphere packings

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Contents

Review

K. Bezdek and S. Reid. Contact graphs of unit sphere packings revisited. *J. Geom.*, 104(1): 57–83, 2013.

K. Bezdek and M. A. Khan. Contact numbers for sphere packings, arXiv:1601.00145.

P. Hlinený and J. Kratochvíl. Representing graphs by disks and balls (a survey of recognition-complexity results). *Discrete Math.*, 229 (2001), 101–124

O. R. Musin. Graphs and spherical two-distance sets. arXiv:1608.03392.

O. R. Musin. Representing graphs by congruent sphere packings, in preparation

O. R. Musin. Analogs of Steiner's porism and Soddy's hexlet in higher dimensions via spherical codes, in preparation

Two–distance sets

A set S in Euclidean space $\mathbb{R}^n$ is called a *two-distance set*, if there are two distances a and b, and the distances between pairs of points of S are either a or b.

If a two-distance set S lies in the unit sphere $\mathbb{S}^{n-1}$, then S is called *spherical two-distance set.*

Euclidean representation of graphs

Let G be a graph on n vertices. Consider a *Euclidean representation of* G in $\mathbb{R}^d$ as a two distance set. In other words, there are two positive real numbers a and b with $b \geq a > 0$ and an embedding f of the vertex set of G into $\mathbb{R}^d$ such that

$$\mathrm{dist}(f(u), f(v)) := \begin{cases} a & \text{if } uv \text{ is an edge of } G \\ b & \text{otherwise} \end{cases}$$

We will call the smallest d such that G is representable in $\mathbb{R}^d$ the *Euclidean representation number* of G and denote it $\mathrm{Erep}(G)$.

Euclidean representation number of graphs

A complete graph K_n represents the edges of a regular $(n-1)$–simplex. So we have $\mathrm{Erep}(\mathrm{K}_n) = n - 1$. That implies

$$\mathrm{Erep}(G) \leq n - 1$$

for any graph G on n vertices.

Since for a two–distance set of cardinality n in $\mathbb{R}^d$

$$n \leq \frac{(d+1)(d+2)}{2}.$$

we have

$$\mathrm{Erep}(G) \geq \frac{\sqrt{8n+1}-3}{2}.$$

Einhorn and Schoenberg work

Einhorn and Schoenberg (ES66) proved that

Theorem

Let G be a simple graph on n vertices. Then $\mathrm{Erep}(G) = n - 1$ if and only if G is a disjoint union of cliques.

Einhorn and Schoenberg work on two–distance sets (1966)

Denote by Σ_n the number of all two–distance sets with n vertices in $\mathbb{R}^{n-2}$. Then

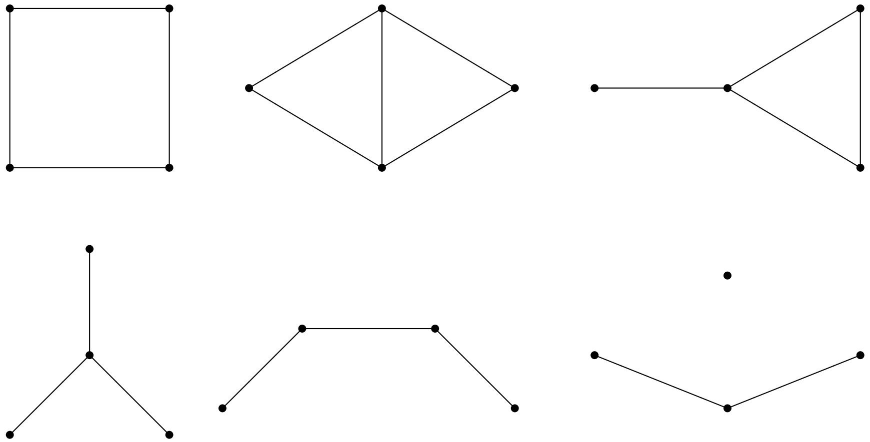
$$\Sigma_n = \Gamma_n - p(n),$$

where Γ_n is the number of all simple undirected graphs and $p(n)$ is the number of unrestricted partitions of n.

$$|\Gamma_4| = 11, \qquad |\Gamma_5| = 34, \qquad |\Gamma_6| = 156, \qquad |\Gamma_7| = 1044, \ldots$$

$$p(4) = 5, \qquad p(5) = 7, \qquad p(6) = 11\ , \qquad p(7) = 15, \ldots$$

$$|\Sigma_4| = 6, \qquad |\Sigma_5| = 27, \qquad |\Sigma_6| = 145, \qquad |\Sigma_7| = 1029, \ldots$$



Let $S = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^{n-1}$. Denote $d_{ij} := \mathrm{dist}(p_i, p_j)$. Consider the Cayley–Menger determinant

$$C_S := \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & d_{12}^2 & \ldots & d_{1n}^2 \\ 1 & d_{21}^2 & 0 & \ldots & d_{2n}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & d_{n1}^2 & d_{n2}^2 & \ldots & 0 \end{vmatrix}$$

Let S be a two-distance set with $a = 1$ and $b > 1$. Then for $i \neq j$,

$$d_{ij}^2 = 1 \quad \text{or} \quad d_{ij}^2 = b^2$$

C_S is a polynomial in $t = b^2$.

Denote this polynomial by $C(t)$.

$$V_{n-1}^2(S) = \frac{(-1)^n \, C_s}{2^{n-1}\,((n-1)!)^2}$$

Actually, Einhorn and Schoenberg considered the discriminating polynomial $D(t)$ that can be defined through the Gram determinant. It is known that

$$C(t) = (-1)^n D(t)$$

Let G be a simple graph. Then

$$C_G(t) := C(t)$$

is uniquely defined by G.

Suppose there is a solution $t > 1$ of $C_G(t) = 0$.

Definition

Denote by τ_1 the smallest root t of C_G such that $t > 1$.

$\mu(G)$ denote the multiplicity of the root τ_1.

If for all roots t of C_G we have $t \leq 1$, then we assume that $\mu(G) := 0$.

The graph complement of G

If $\mu(G) > 0$, then $\tau_0(G) := 1/\tau_1(G)$ is a root of $C_{\bar{G}}(t)$ and $\tau_1(\bar{G}) = 1/\tau_0(G)$. Note that there are no more roots of $C_G(t)$ on the interval $[\tau_0(G), \tau_1(G)]$.

$C_{\bar{G}}(t)$ is the reciprocal polynomial of $C_G(t)$, i.e.

$$C_{\bar{G}}(t) = t^k C_G(1/t), \quad k = \deg C_G(t).$$

The Einhorn–Schoenberg theorem

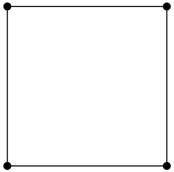
Theorem

Let G be a simple graph on n vertices. Then

$$\mathrm{Erep}(G) = n - \mu(G) - 1$$

If $\mu(G) > 0$, then a minimal Euclidean representation of G is uniquely define up to isometry.

$$C_1(t) = t^2(2 - t), \quad C_2(t) = t(3 - t), \quad C_3(t) = -t^2 + 4t - 1$$



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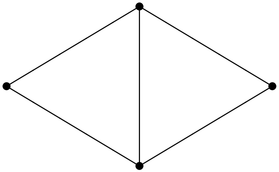

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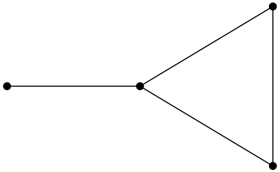

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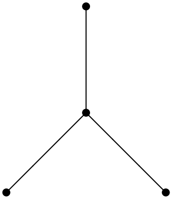
$$C_4(t) = t^2(3-t), \quad C_5(t) = (t+1)(3t-t^2-1), \quad C_6(t) = -t^2+4t-1$$



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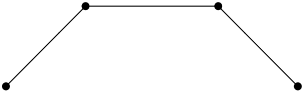

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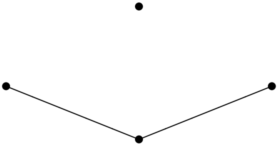

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$G = K_{2,\ldots,2}$

Theorem

Let G be a complete m–partite graph $K_{2,\ldots,2}$. Then $\mathrm{Erep}(G) = m$ and a minimal Euclidean representation of G is a regular cross–polytope.

Proof.

We have $n = 2m$ and

$$C_G(t) = 2m\, t^m (2 - t)^{m-1}.$$

Then $\tau_1 = 2$ and $\mu(G) = m - 1$. Thus, $\mathrm{Erep}(K_{2,\ldots,2}) = m$. □

V. Alexandrov (2016)

$G = K_{2,\dots,2}$: geometric proof

Lemma

Let for sets X_1 and X_2 in $\mathbb{R}^d$ there is $a > 0$ such that $\mathrm{dist}(p_1, p_2) = a$ for all $p_1 \in X_1$, $p_2 \in X_2$.
Then both X_i are spherical sets and the affine spans $\mathrm{aff}(X_i)$ in $\mathbb{R}^d$ are orthogonal each other.

Let $S := f(V(G))$ in $\mathbb{R}^d$. Then $\mathbb{R}^d$ can be split into the orthogonal product $\prod_{i=1}^{m} L_i$ of lines such that for $S_i := S \cap L_i$ we have $|S_i| = 2$. Thus, $d = m$ and S is a regular cross–polytope.

Spherical representations of graphs

Let f be a Euclidean representation of a graph G on n vertices in $\mathbb{R}^d$ as a two distance set. We say that f is a *spherical representation of* G if the image $f(G)$ lies on a $(d-1)$–sphere in $\mathbb{R}^d$. We will call the smallest d such that G is spherically representable in $\mathbb{R}^d$ the *spherical representation number* of G and denote it $\mathrm{Srep}(G)$.

Nozaki and Shinohara (2012) using Roy's results (2010) give a necessary and sufficient condition of a Euclidean representation of a graph G to be spherical.

We define a polynomial $M_G(t)$ and show that a Euclidean representation is spherical if and only if the multiplicity of $\tau_1(G)$ is the same for $C_G(t)$ and $M_G(t)$

Spherical representations of graphs

Let $S = \{p_1, \dots, p_n\}$ be a set in $\mathbb{R}^{n-1}$. As above $d_{ij} := \mathrm{dist}(p_i, p_j)$. Let

$$M_S := \begin{vmatrix} 0 & d_{12}^2 & \dots & d_{1n}^2 \\ d_{21}^2 & 0 & \dots & d_{2n}^2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ d_{n1}^2 & d_{n2}^2 & \dots & 0 \end{vmatrix}$$

The circumradius of a simplex

It is well known, that if the points in S form a simplex of dimension $(n-1)$, the radius R of the sphere circumscribed around this simplex is given by

$$R^2 = -\frac{1}{2}\frac{M_S}{C_S}.$$

Spherical representations of graphs

For a given graph G we denote by $M_G(t)$ the polynomial in $t=b^2$ that defined by M_S. Let

$$F_G(t) := -\frac{1}{2}\frac{M_G(t)}{C_G(t)}.$$

If G is a graph with $\mu(G)>0$ and $F_G(\tau_1)<\infty$, then denote $\mathcal{R}(G) := \sqrt{F_G(\tau_1)}$. Otherwise, put $\mathcal{R}(G) := \infty$.

We will call $\mathcal{R}(G)$ the circumradius of G.

Spherical representations of graphs

Theorem

Let G be a graph on n vertices with $\mathcal{R}(G) < \infty$. Then $\mathrm{Srep}(G) = n - \mu(G) - 1$, *otherwise* $\mathrm{Srep}(G) = n - 1$.

The circumradius of a graph

Theorem

$\mathcal{R}(G) \geq 1/\sqrt{2}$.

It is not clear what is the range of $\mathcal{R}(G)$? If $\mathcal{R}(G) < \infty$, then for a fixed n there are only finitely many cases. Thus the range is a countable set.

Open question. *Suppose $\mathcal{R}(G) < \infty$. What is the upper bound of $\mathcal{R}(G)$? Can $\mathcal{R}(G)$ be greater than 1?*

J–spherical representation of graphs

We have $\mathcal{R}(G) \geq 1/\sqrt{2}$. Now consider the boundary case $\mathcal{R}(G) = 1/\sqrt{2}$.

Definition

Let f be a spherical representation of a graph G in $\mathbb{R}^d$ as a two distance set. We say that f is a J–spherical representation of G if the image $f(G)$ lies in the unit sphere $\mathbb{S}^{d-1}$ and the first (minimum) distance $a = \sqrt{2}$.

Theorem

For any graph $G \neq K_n$ there is a unique (up to isometry) J–spherical representation.

J–spherical representation of graphs

The uniqueness of a J–spherical representation of $G \neq K_n$ shows that the following definition is correct.

Definition

$\mathrm{Jrep}(G)$ = *J–spherical representation dimension*
$b_*(G)$ = *the second distance of this representation.*

If G is the pentagon, then $\mathrm{Srep}(G) = 2 < \mathrm{Jrep}(G) = 4$.

Theorem

Let $G \neq K_n$ be a graph on n vertices. If $\mathcal{R}(G) = 1/\sqrt{2}$, then

$$\mathrm{Jrep}(G) = n - \mu(G) - 1, \ \ \textit{otherwise} \ \ \mathrm{Jrep}(G) = n - 1.$$

Representation numbers of the join of graphs

Recall that the *join* $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

Representation numbers of the join of graphs

Definition

We say that G on n vertices is J–simple if $\mathrm{Jrep}(G) = n - 1$.

Theorem

Let $G := G_1 + \ldots + G_m$. Suppose all G_i are J–simple and

$$b_*(G_1) = \ldots = b_*(G_k) < b_*(G_{k+1}) \leq \ldots \leq b_*(G_m).$$

Then

$$\mathrm{Jrep}(G) = \mathrm{Srep}(G) = n - k, \;\; \mathrm{Erep}(G) = n - \max(k, 2),$$

where n denote the number of vertices of G.

Representation numbers of complete multipartite graphs

Corollary

Let G be a complete multipartite graph $K_{n_1 \ldots n_m}$. Suppose

$$n_1 = \ldots = n_k > n_{k+1} \geq \ldots \geq n_m.$$

Let $n := n_1 + \ldots + n_m$. Then

1. *If $k = 1$, then $\mathrm{Erep}(G) = n - 2$, otherwise $\mathrm{Erep}(G) = n - k$;*
2. $\mathrm{Srep}(G) = n - k$;
3. $\mathrm{Jrep}(G) = n - k$.

Note that Statement 1 in the Corollary first proved by Roy (2010).

Contact graph

Let X be a finite subset of a metric space M. Denote

$$\psi(X) := \min_{x,y \in X} \{\mathrm{dist}(x,y)\}, \text{ where } x \neq y.$$

The *contact graph* $\mathrm{CG}(X)$ is a graph with vertices in X and edges (x, y), $x, y \in X$, such that $\mathrm{dist}(x, y) = \psi(X)$.
In other words, $\mathrm{CG}(X)$ is the contact graph of a packing of spheres of diameter $\psi(X)$ with centers in X.

Euclidean representations

$M = \mathbb{R}^d$ and $M = \mathbb{S}^{d-1}$. Let $G = (V, E)$ be a simple graph with at least one edge. Let $f : V \to \mathbb{R}^d$ be a minimal Euclidean contact graph representation. Then denote the dimension d by $\mathrm{dim}_{\mathrm{E}}(G)$.

Theorem

Let G be a graph on n vertices. Let $G \neq K_n$. Then

$$\mathrm{dim}_{\mathrm{E}}(G) \leq n - 2.$$

Spherical representations

Let X be a spherical representation of G in $\mathbb{S}^{d-1}$, i.e. $\mathrm{CG}(X) = G$. Denote by $\mathrm{dim}_{\mathrm{S}}(G, \theta)$ the smallest dimension d such that $\psi(X) = \theta$. The dimension of a minimal spherical contact graph representation of G we denote $\mathrm{dim}_{\mathrm{S}}(G)$,

$$\mathrm{dim}_{\mathrm{S}}(G) := \min_{0<\theta<\theta_0} \mathrm{dim}_{\mathrm{S}}(G, \theta), \ \theta_0 := \arccos(-1/(n-1)).$$

Theorem

Let $G = (V, E)$ be a graph on n vertices. Let $0 < \theta < \theta_0$. Then

$$\mathrm{dim}_{\mathrm{S}}(G, \theta) \leq n - 1.$$

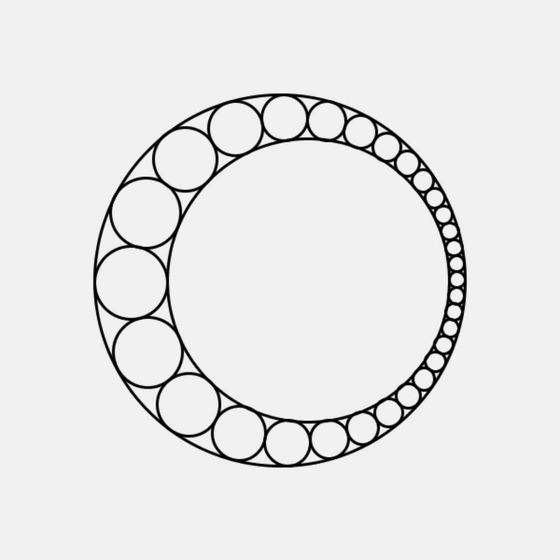
Join of graphs

The orthogonality lemma implies explicit formulas for the graph join and multipartite graphs $K_{n_1 \dots n_m}$.

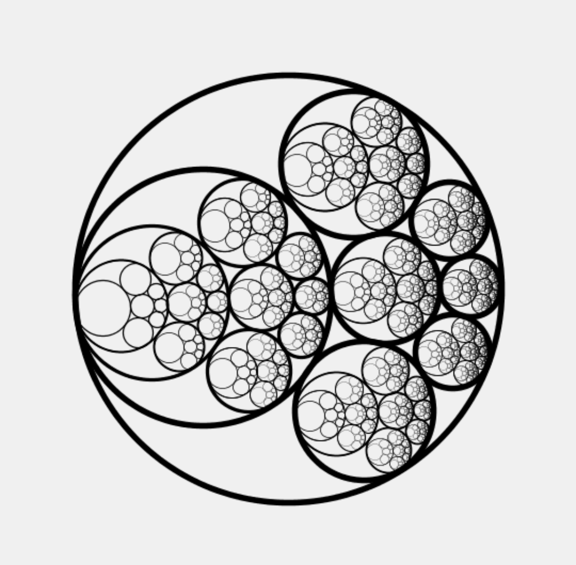
Steiner's porism

If a Steiner chain is formed from one starting circle, then a Steiner chain is formed from any other starting circle.
G6bor Dam6sdi

Steiner's chain



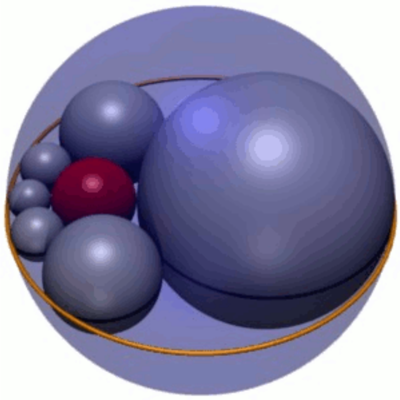
Steiner's chain



Soddy's hexlet

Soddy's hexlet is a chain of six spheres each of which is tangent to both of its neighbors and also to three mutually tangent given spheres.

Soddy's hexlet



Inversion T

Let S_1 and S_2 be spheres in $\mathbb{R}^n$. Consider two cases:
(i) S_1 and S_2 are tangent;
(ii) S_1 and S_2 do not touch each other.
In case (i) let O be the contact point of these spheres and if we apply the sphere inversion T with center O and an arbitrary radius ρ, then S_1 and S_2 become two parallel hyperplanes S_1' and S_2'.
In case (ii) we can use the famous theorem: *There is T that invert S_1 and S_2 into a pair of concentric spheres S_1' and S_2'.*

Lemma

The radius r_T of $S' = T(S)$ is the same for all spheres S that are tangent to S_1 and S_2.

$\mathcal{F}$–kissing arrangements and spherical codes

Let $\mathcal{F} = \{S_1, \dots, S_m\}$, $2 \leq m < n + 2$, be a family of m spheres in $\mathbb{R}^n$ such that S_1 and S_2 are non–intersecting or tangent spheres.
We say that a set $\mathcal{C}$ of spheres in $\mathbb{R}^n$ is an $\mathcal{F}$–kissing arrangement if
(1) each sphere from $\mathcal{C}$ is tangent all spheres from $\mathcal{F}$,
(2) any two distinct spheres from $\mathcal{C}$ are non–intersecting.

Theorem

For a given $\mathcal{F}$ the inversion T defines a one–to–one correspondence between $\mathcal{F}$–kissing arrangements and spherical $\psi_{\mathcal{F}}$–codes in $\mathbb{S}^{d-1}$, where $\psi_{\mathcal{F}} \in [0, \infty]$ is uniquely defined by $\mathcal{F}$.

Analogs of Steiner's porism

Theorem

Let $\mathcal{F} = \{S_1, \ldots, S_m\}$, $2 \leq m < n + 2$, *be a family of m spheres in* $\mathbb{R}^n$ *such that* S_1 *and* S_2 *are non–intersecting spheres. If a Steiner packing is formed from one starting sphere, then a Steiner packing is formed from any other starting packing.*

Steiner's packings

Proposition

If for a family $\mathcal{F}$ there exist a simplicial $\mathcal{F}$–kissing arrangement then we have one of the following cases

1. $d = 2$, $\psi_{\mathcal{F}} = 2\pi/k$, $k \geq 3$, *and* $P_{\mathcal{F}}$ *is a regular polygon with* k *vertices.*
2. $\psi_{\mathcal{F}} = \arccos(-1/d)$ *and* $P_{\mathcal{F}}$ *is a regular* d*–simplex with any* $d \geq 2$*.*
3. $\psi_{\mathcal{F}} = \pi/2$ *and* $P_{\mathcal{F}}$ *is a regular* d*–crosspolytope with any* $d \geq 2$*.*
4. $d = 3$, $\psi_{\mathcal{F}} = \arccos(1/\sqrt{5})$ *and* $P_{\mathcal{F}}$ *is a regular icosahedron.*
5. $d = 4$, $\psi_{\mathcal{F}} = \pi/5$ *and* $P_{\mathcal{F}}$ *is a regular 600–cell.*

Analogs of Soddy's hexlet

Theorem

Let $3 \leq m < n + 2$. Let X be a spherical ψ_m–codes in $\mathbb{S}^{d-1}$, where $d := n + 2 - m$. Then for any family $\mathcal{F}$ of m mutually tangent spheres in $\mathbb{R}^n$ there is an $\mathcal{F}$–kissing arrangement that is correspondent to X.

THANK YOU