On the long time stability of a temporal discretization scheme for the three dimensional primitive equations

Chun-Hsiung Hsia

National Taiwan University

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1. 1922 Richardson
2. 1992 Lions, Temam and Wang
3. 2007 Kobelkov; Cao and Titi
4. 2007 Ju

The Primitive Equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + w\frac{\partial \boldsymbol{v}}{\partial z} + f\boldsymbol{v}^{\perp} + \nabla p = \nu_1 \triangle \boldsymbol{v} + \mu_1 \frac{\partial^2 \boldsymbol{v}}{\partial z^2} + F_1$$

$$\frac{\partial p}{\partial z} = -\theta$$

$$\nabla \cdot \boldsymbol{v} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{v} \cdot \nabla)\theta + w\frac{\partial \theta}{\partial z} = \nu_2 \triangle \theta + \mu_2 \frac{\partial^2 \theta}{\partial z^2} + F_2$$

Cylinderical domain

$$\mathcal{M} = \mathcal{M}' \times (-h, 0),$$

Boundary Condition

$$\Gamma_u : \frac{\partial \boldsymbol{v}}{\partial z} = 0,\ w = 0,\ \frac{\partial \theta}{\partial z} = 0,$$

$$\Gamma_b : \frac{\partial \boldsymbol{v}}{\partial z} = 0,\ w = 0,\ \frac{\partial \theta}{\partial z} = 0,$$

$$\Gamma_l : \boldsymbol{v} \cdot \vec{n} = 0,\ \frac{\partial \boldsymbol{v}}{\partial \vec{n}} \times \vec{n} = 0, \frac{\partial \theta}{\partial \vec{n}} = 0.$$

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$$
\begin{aligned}
&H = H_1 \times H_2, \quad V = V_1 \times V_2,\\
&H_1 = \left\{ \boldsymbol{v} \in (L^2(\mathcal{M}))^2 : \int_{-h}^{0} \nabla \cdot \boldsymbol{v}\, dz = 0, \quad \boldsymbol{v} \cdot \vec{n} = 0, \text{ on } \Gamma_l \right\},\\
&V_1 = \left\{ \boldsymbol{v} \in (H^1(\mathcal{M}))^2 : \int_{-h}^{0} \nabla \cdot \boldsymbol{v}\, dz = 0, \quad \boldsymbol{v} \cdot \vec{n} = 0, \text{ on } \Gamma_l \right\},\\
&H_2 = L^2(\mathcal{M}), \quad V_2 = H^1(\mathcal{M}).
\end{aligned}
$$

Cao-Titi, Kobelkov

Let $F_1 = 0$, $F_2 \in H^1(\mathcal{M})$, $(\boldsymbol{v}_0, \theta_0) \in V_1 \times V_2$ and $T > 0$, then there exists a unique strong solution $(\boldsymbol{v}, \theta)$ to the system of 3D viscous Primitive equations on the interval $[0, T]$, which depends on the initial data continuously in $H_1 \times H_2$.

Reformulation of w and p

$$w(x,y,z,t) = -\int_{-h}^{z} \nabla \cdot \boldsymbol{v}(x,y,\xi,t)\, d\xi$$

$$p(x,y,z,t) = p_0(x,y,t) - \int_{-h}^{z} \theta(x,y,\xi,t)\, d\xi$$

Integral differential equation

$$
\begin{aligned}
&\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \Big(\int_{-h}^{z} \nabla\cdot\boldsymbol{v}(x,y,\xi,t)\,d\xi\Big)\frac{\partial \boldsymbol{v}}{\partial z} + f\boldsymbol{v}^{\perp} \\
&+\nabla p_0 - \nabla\Big(\int_{-h}^{z}\theta(x,y,\xi,t)\,d\xi\Big) = \nu_1\triangle\boldsymbol{v} + \mu_1\frac{\partial^2\boldsymbol{v}}{\partial z^2} + F_1, \\
&\frac{\partial\theta}{\partial t} + (\boldsymbol{v}\cdot\nabla)\theta - \Big(\int_{-h}^{z}\nabla\cdot\boldsymbol{v}(x,y,\xi,t)\,d\xi\Big)\frac{\partial\theta}{\partial z} \\
&= \nu_2\triangle\theta + \mu_2\frac{\partial^2\theta}{\partial z^2} + F_2, \\
&\frac{\partial\boldsymbol{v}}{\partial z}\Big|_{\Gamma_u} = \frac{\partial\boldsymbol{v}}{\partial z}\Big|_{\Gamma_b} = 0,\ \boldsymbol{v}\cdot\vec{n}\Big|_{\Gamma_l} = 0,\ \frac{\partial\boldsymbol{v}}{\partial\vec{n}}\times\vec{n}\Big|_{\Gamma_l} = 0, \\
&\frac{\partial\theta}{\partial z}\Big|_{\Gamma_u} = \frac{\partial\theta}{\partial z}\Big|_{\Gamma_b} = 0,\ \frac{\partial\theta}{\partial\vec{n}}\Big|_{\Gamma_l} = 0.
\end{aligned}
$$

$$\bar{\boldsymbol{v}} = \frac{1}{h}\int_{-h}^{0} \boldsymbol{v}(x, y, z, t)\, dz,$$

$$\tilde{\boldsymbol{v}} = \boldsymbol{v} - \bar{\boldsymbol{v}}.$$

Average

$$\frac{\partial \bar{\boldsymbol{v}}}{\partial t} + (\bar{\boldsymbol{v}} \cdot \nabla)\bar{\boldsymbol{v}} + \overline{(\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{v}}} + \overline{(\nabla \cdot \tilde{\boldsymbol{v}})\tilde{\boldsymbol{v}}} + f\bar{\boldsymbol{v}}^{\perp}$$

$$+\nabla p_0 - \overline{\int_{-h}^{z} \nabla\theta(x, y, \xi, t)\, d\xi} = \nu_1 \triangle \bar{\boldsymbol{v}} + \bar{F}_1,$$

$$\nabla \cdot \bar{\boldsymbol{v}} = 0,$$

$$\bar{\boldsymbol{v}} \cdot \vec{n}|_{\partial \mathcal{M}'} = 0, \quad \frac{\partial \bar{\boldsymbol{v}}}{\partial \vec{n}} \times \vec{n}|_{\partial \mathcal{M}'} = 0,$$

Perturbation

$$\frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{v}} - \Big(\int_{-h}^{z} \nabla \cdot \tilde{\boldsymbol{v}}(x,y,\xi,t)\, d\xi\Big) \frac{\partial \tilde{\boldsymbol{v}}}{\partial z}$$

$$+(\tilde{\boldsymbol{v}} \cdot \nabla)\bar{\boldsymbol{v}} + (\bar{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{v}} - \overline{(\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{v}} + (\nabla \cdot \tilde{\boldsymbol{v}})\tilde{\boldsymbol{v}}} + f\tilde{\boldsymbol{v}}^{\perp}$$

$$- \int_{-h}^{z} \nabla\theta(x,y,\xi,t)\, d\xi + \overline{\int_{-h}^{z} \nabla\theta(x,y,\xi,t)\, d\xi}$$

$$= \nu_1 \triangle \tilde{\boldsymbol{v}} + \mu_1 \frac{\partial^2 \tilde{\boldsymbol{v}}}{\partial z^2} + \tilde{F}_1,$$

$$\frac{\partial \tilde{\boldsymbol{v}}}{\partial z}\Big|_{\Gamma_u} = \frac{\partial \tilde{\boldsymbol{v}}}{\partial z}\Big|_{\Gamma_b} = 0,\ \tilde{\boldsymbol{v}} \cdot \vec{n}\Big|_{\partial\Gamma_l} = 0, \quad \frac{\partial \tilde{\boldsymbol{v}}}{\partial \vec{n}} \times \vec{n}\Big|_{\partial\Gamma_l} = 0,$$

Let $u_i = (\boldsymbol{v}_i, \sqrt{\beta}\theta_i)$, $i = 1, 2, 3$. We define the bilinear form

$$a(u_1, u_2) = a_1(u_1, u_2) + a_2(u_1, u_2), \tag{1}$$

where

$$a_1(u_1, u_2) = \nu \int_{\mathcal{M}} \nabla_3 \boldsymbol{v}_1 \cdot \nabla_3 \boldsymbol{v}_2 d\mathcal{M} + \int_{\mathcal{M}} (\int_{-h}^{z} \theta_1 d\xi)(\nabla_2 \cdot \boldsymbol{v}_2) d\mathcal{M}, \tag{2}$$

$$a_2(u_1, u_2) = \beta\nu \int_{\mathcal{M}} (\nabla_3 \theta_1 \cdot \nabla_3 \theta_2) d\mathcal{M} + \beta\nu\alpha \int_{\Gamma_u} \theta_1 \theta_2 d\Gamma_u, \tag{3}$$

and $\beta = h^2/\nu^2 + 1$ is a positive constant.

The trilinear form b is defined by

$$b(u_1, u_2, u_3) = b_1(u_1, u_2, u_3) + \beta b_2(u_1, u_2, u_3), \tag{4}$$

where

$$b_1(u_1, u_2, u_3) = \int_{\mathcal{M}} \left((\boldsymbol{v}_1 \cdot \nabla_2)\boldsymbol{v}_2 + w(\boldsymbol{v}_1)\frac{\partial \boldsymbol{v}_2}{\partial z} \right) \cdot \boldsymbol{v}_3 d\mathcal{M}, \tag{5}$$

$$b_2(u_1, u_2, u_3) = \int_{\mathcal{M}} \left((\boldsymbol{v}_1 \cdot \nabla_2)\theta_2 + w(\boldsymbol{v}_1)\frac{\partial \theta_2}{\partial z} \right) \theta_3 d\mathcal{M}. \tag{6}$$

Let the operators A and B satisfy

$$
\begin{aligned}
&< Au_1, u_2 >= a(u_1, u_2), \\
&< B(u_1, u_2), u_3 >= b(u_1, u_2, u_3).
\end{aligned} \tag{7}
$$

A time discretized semi-implicit Euler scheme of the primitive equation is given by

$$
\frac{u^n - u^{n-1}}{k} + Au^n + B(u^{n-1}, u^n) + Eu^n = F^n, \tag{8}
$$

with initial condition

$$
u^0 = u_0,
$$

where

$$
\begin{aligned}
&u^n = (\boldsymbol{v}^n, \sqrt{\beta}\theta^n) = (\boldsymbol{v}(x,y,z,nk), \beta\theta(x,y,z,nk)), \\
&F^n = \frac{1}{k}\int_{(n-1)k}^{nk} F(x,y,z,t)dt, \\
&F = (F_1, \sqrt{\beta}F_2).
\end{aligned}
$$

Goal: To show that the H^1 norm of u^n is bounded for all n $\in \mathbb{N}$ as long as the time step k is less than certain threshold.

Geophysical Fluid Dynamics

1. Oceanic fluid is made up of a slightly compressible fluid with Coriolis force.
2. Often described by Navier-Stokes equation, Boussinesq equation and primitive equation.
3. Important characteristics : stratification, rotation, temporal periodicity, steady states
4. Only the stable flows could be observed in real world or numerical experiments.

Time periodic flows

(A) (Bona-Hsia-Ma-Wang, 2011) The Hopf bifurcation diagrams of the double-diffusive equation (Boussinesq equation coupled with temperature diffusion and salinity diffusion) is clearly classified according to the regions in the phase space of temperature Rayleigh number and salinity Rayleigh number under suitable physical conditions. This demonstrates a mechanism that produces time periodic circulations due to stratification.

(B) (Hsia-Shiue, 2013; Hsia, Jung, Kwon, Nguyen, Chen, Shiue) The analysis for Navier-Stokes equations and viscous Burgers' equations demonstrates the existence of time periodic flows due to the time periodic external force. The rigorous mathematical analysis shows that there exists at least one (stable or unstable) time periodic flow with the presence of time periodic force in the GFD system.

The effects of time periodic force in GFD

Let f denote the amplitude of the external force in certain appropriate norm. There exists two positive numbers $0 < f_1 < f_2$ such that

1. For $0 < f < f_1$, the time periodic flow is temporal asymptotically stable (Hence, the time periodic flow is unique.).
2. In case $f_1 < f < f_2$, the numerical experiments show that there exist several locally temporal asymptotically stable time periodic flows.
3. If $f > f_2$, the numerical experiments show that there does not exist stable flows.

Physical Implications

Intuitively speaking, it is reasonable to expect that the time periodic external force produces the time periodic flows. This is verified by rigorous mathematics in our analysis for a wide class of model equations. However, while the force is too large ($f > f_2$), the time periodic flows lose its stabilities which means it cannot be captured by physical or numerical experiments. Only the flows generated by small time periodic force ($f < f_2$)could be observed.

1. L^2 estimates for $\boldsymbol{v}^n$ and θ^n.
2. L^4 estimates for θ^n.
3. L^4 estimates for $\tilde{\boldsymbol{v}}^n$.
4. L^{12} estimates for $\tilde{\boldsymbol{v}}^n$.
5. L^4 estimates for p_0^n.
6. L^4 estimates for $\bar{\boldsymbol{v}}^n$.
7. L^{12} estimates for $\bar{\boldsymbol{v}}^n$.
8. L^2 estimates for $\frac{\partial \boldsymbol{v}^n}{\partial z}$.
9. L^4 estimates for $\frac{\partial \boldsymbol{v}^n}{\partial z}$.
10. L^2 estimates for $\nabla_2 \boldsymbol{v}^n$.
11. L^2 estimates for $\frac{\partial \theta^n}{\partial z}$.
12. L^2 estimates for $\nabla_2 \theta^n$.

Discrete Gronwall Lemma

For given $k > 0$ and a positive integer $n_1 > 1$, suppose the positive sequences ξ_n, η_n and χ_n satisfy

$$\xi_n \leq \xi_{n-1}(1 + k\eta_{n-1}) + k\chi_n, \qquad \text{for } n = 1, \cdots, n_1. \tag{9}$$

Then we have, for any $n \in \{2, \cdots, n_1\}$,

$$\xi_n \leq \xi_0 \exp(\sum_{i=0}^{n-1} k\eta_i) + \sum_{i=1}^{n-1} k\chi_i \exp(\sum_{l=i}^{n-1} k\eta_l) + k\chi_n. \tag{10}$$

Discrete Uniform Gronwall Lemma

Given $k > 0$, positive integers n_1, n_2, m with $n_1 + n_2 + 1 \leq m$, suppose the positive sequences ξ_n, η_n and χ_n satisfy

$$\xi_n \leq \xi_{n-1}(1 + k\eta_{n-1}) + k\chi_n, \qquad \text{for } n = n_1, n_1 + 1, \cdots, m, \quad (11)$$

and there exist positive numbers a_1, a_2 and a_3 such that

$$\sum_{n=j}^{j+n_2} k\eta_n \leq a_1, \qquad \sum_{n=j}^{j+n_2} k\chi_n \leq a_2, \qquad \sum_{n=j}^{j+n_2} k\xi_n \leq a_3, \quad (12)$$

for any j with $n_1 \leq j \leq m - n_2$. Then, we have

$$\xi_n \leq \left(\frac{a_3}{kn_2} + a_2\right)e^{a_1}, \quad (13)$$

for any n with $n_1 + n_2 + 1 \leq n \leq m$.

Sobolev and Ladyzhenskaya's inequalities in $\mathbb{R}^3$

For $\phi \in H^1(\mathcal{M})$, one has

$$|\phi|_{L^3(\mathcal{M})} \leq C_0 |\phi|^{1/2}_{L^2(\mathcal{M})} |\phi|^{1/2}_{H^1(\mathcal{M})}, \tag{14}$$

$$|\phi|_{L^4(\mathcal{M})} \leq C_0 |\phi|^{1/4}_{L^2(\mathcal{M})} |\phi|^{3/4}_{H^1(\mathcal{M})}, \tag{15}$$

$$|\phi|_{L^6(\mathcal{M})} \leq C_0 |\phi|_{H^1(\mathcal{M})}. \tag{16}$$

Taking the $L^2(\mathcal{M})$ inner product of (8) with $2k(\boldsymbol{v}^n, \theta^n)$,

$$\|u^n\|^2 - \|u^{n-1}\|^2 + \|u^n - u^{n-1}\|^2 + 2ka(u^n, u^n) = 2k < F^n, u^n >, \tag{17}$$

where

$$\|u\|^2 := \|\boldsymbol{v}\|^2 + \beta\|\theta\|^2, \qquad \text{for } u = (\boldsymbol{v}, \theta).$$

By Poincare inequality, there exists a positive constant λ such that

$$\begin{aligned} a(u,u) &\geq \frac{1}{2}\nu\|\nabla_3 \boldsymbol{v}\|^2 + \frac{1}{2}\beta\nu\|\nabla_3\theta\|^2 + \beta\nu\alpha\int_{\Gamma_u}\theta^2 d\Gamma_u \\ &\geq \frac{1}{2}\nu\lambda\|u\|^2. \end{aligned} \tag{18}$$

On the other hand, we have

$$|2k < F^n, u^n > | \leq \frac{1}{2}k\nu\lambda\|u^n\|^2 + \frac{2k}{\nu\lambda}\|F^n\|^2 \leq \frac{1}{2}k\nu\lambda\|u^n\|^2 + \frac{2k}{\nu\lambda}\|F\|_\infty^2, \tag{19}$$

where

$$\|F^n\|^2 = \|F_1^n\|^2 + \beta\|F_2^n\|^2.$$

By (18) and (19), we then derive from (17),

$$\begin{aligned}\|u^n\|^2 - \|u^{n-1}\|^2 + \|u^n - u^{n-1}\|^2 + \frac{1}{2}k\nu\lambda\|u^n\|^2 \\ + \frac{1}{2}k\nu\|\nabla_3 \boldsymbol{v}^n\|^2 + \frac{1}{2}\beta k\nu\|\nabla_3\theta^n\|^2 \\ + \beta k\nu\alpha\int_{\Gamma_u}(\theta^n)^2 d\Gamma_u \\ \leq \frac{1}{2}k\nu\lambda\|u^n\|^2 + \frac{2k}{\nu\lambda}\|F\|_\infty^2 \end{aligned} \tag{20}$$

and hence,

$$(1+\frac{1}{2}k\nu\lambda)\|u^n\|^2 \leq \|u^{n-1}\|^2 + \frac{2k}{\nu\lambda}\|F\|_\infty^2. \tag{21}$$

Inductively, we obtain that

$$\|u^n\|^2 \leq (1+\frac{1}{2}k\nu\lambda)^{-n}\|u_0\|^2 + \left(1-(1+\frac{1}{2}k\nu\lambda)^{-n}\right)\frac{4\|F\|_\infty^2}{\nu^2\lambda^2}. \quad (22)$$

Summing up (20) over n from j to $j + n_2$, by (22), we obtain

$$\sum_{n=j}^{j+n_2} \Big(\frac{1}{2}k\nu\|\nabla_3 \boldsymbol{v}^n\|^2 + \frac{1}{2}\beta k\nu\|\nabla_3\theta^n\|^2 + \beta k\nu\alpha\int_{\Gamma_u}(\theta^n)^2 d\Gamma_u\Big)$$
$$\le (1 + \frac{1}{2}k\nu\lambda)^{-(j-1)}\|u_0\|^2$$
$$+ \Big(1 - (1 + \frac{1}{2}k\nu\lambda)^{-(j-1)}\Big)\frac{4\|F\|_\infty^2}{\nu^2\lambda^2} + \frac{2k(n_2+1)}{\nu\lambda}\|F\|_\infty^2. \tag{23}$$

L^4 Estimates

Taking the $L^2(\mathcal{M})$ inner product of $4k(\theta^n)^3$ with the equation

$$\frac{\theta^n - \theta^{n-1}}{k} + (\boldsymbol{v}^{n-1} \cdot \nabla_2)\theta^n - \Big(\int_{-h}^{z} \nabla_2 \cdot \boldsymbol{v}^{n-1}(x, y, \xi, t)\, d\xi\Big)\frac{\partial \theta^n}{\partial z}$$
$$= \nu \triangle_3 \theta^n + F_2^n,$$

Use the facts

$$4(\theta^n - \theta^{n-1})(\theta^n)^3 = 4|\theta^n|^4 - 2|\theta^n|^2(|\theta^n|^2 + |\theta^{n-1}|^2 - |\theta^n - \theta^{n-1}|^2)$$
$$\geq |\theta^n|^4 - |\theta^{n-1}|^4 + 2|\theta^n|^2|\theta^n - \theta^{n-1}|^2,$$

and

$$\||\theta^n|^3\| = \||\theta^n|^2\|_{L^3(\mathcal{M})}^{\frac{3}{2}}$$
$$\leq \left(C_0\||\theta^n|^2\|^{\frac{1}{2}}\|\nabla_3|\theta^n|^2\|^{\frac{1}{2}}\right)^{\frac{3}{2}}$$
$$\leq 2C_0^{\frac{3}{2}}\|\theta^n\|_{L^4(\mathcal{M})}^{\frac{3}{2}}(\int_{\mathcal{M}}|\theta^n|^2|\nabla_3\theta^n|^2 d\mathcal{M})^{\frac{3}{8}}.$$

$$
\begin{aligned}
&\|\theta^n\|^4_{L^4(\mathcal{M})} - \|\theta^{n-1}\|^4_{L^4(\mathcal{M})} + 2\int_{\mathcal{M}} |\theta^n|^2|\theta^n - \theta^{n-1}|^2 d\mathcal{M} \\
&\quad + 12k\nu \int_{\mathcal{M}} |\theta^n|^2|\nabla_3\theta^n|^2 d\mathcal{M} + 4k\nu\alpha \int_{\Gamma_u} |\theta^n|^4 d\mathcal{M}' \\
&\quad \leq 4k\|F_2^n\|\||\theta^n|^3\|_{L^2} \\
&\quad \leq 8kC_0^{\frac{3}{2}}\|F_2^n\|\|\theta^n\|^{\frac{3}{2}}_{L^4(\mathcal{M})}\left(\int_{\mathcal{M}} |\theta^n|^2|\nabla_3\theta^n|^2 d\mathcal{M}\right)^{\frac{3}{8}} \\
&\quad \leq \frac{2^{\frac{9}{5}}5C_0^{\frac{12}{5}}k}{\nu^{\frac{3}{5}}}\|F_2^n\|^{\frac{8}{5}}\|\theta^n\|^{\frac{12}{5}}_{L^4(\mathcal{M})} + \frac{3}{8}k\nu \int_{\mathcal{M}} |\theta^n|^2|\nabla_3\theta^n|^2 d\mathcal{M}
\end{aligned}
\tag{24}
$$

If $\|\theta^n\|_{L^4} \neq 0$, inferring from (24), we see that

$$\|\theta^n\|^2_{L^4(\mathcal{M})} \leq \|\theta^{n-1}\|^2_{L^4(\mathcal{M})} + \frac{2^{\frac{9}{5}} 5 C_0^{\frac{12}{5}} k}{\nu^{\frac{3}{5}}} \times \left(\frac{\|\theta^n\|^2_{L^4(\mathcal{M})}}{\|\theta^n\|^2_{L^4(\mathcal{M})} + \|\theta^{n-1}\|^2_{L^4(\mathcal{M})}}\right) \|F_2^n\|^{\frac{8}{5}} \|\theta^n\|^{\frac{2}{5}}_{L^4(\mathcal{M})}$$
$$\leq \|\theta^{n-1}\|^2_{L^4(\mathcal{M})} + \frac{2^{\frac{9}{5}} 5 C_0^{\frac{12}{5}} k}{\nu^{\frac{3}{5}}} \|F_2^n\|^{\frac{8}{5}} \|\theta^n\|^{\frac{2}{5}}_{L^4(\mathcal{M})}$$
$$\leq \|\theta^{n-1}\|^2_{L^4(\mathcal{M})} + \frac{1}{5} k \|\theta^n\|^2_{L^4(\mathcal{M})} + \left(\frac{2^5 5^{\frac{1}{4}} C_0^3}{\nu^{\frac{3}{4}}}\right) k \|F_2^n\|^2. \tag{25}$$

Assuming $k < 2$, we obtain from (25) that

$$\|\theta^n\|_{L^4(\mathcal{M})}^2 \leq 2\|\theta^{n-1}\|_{L^4(\mathcal{M})}^2 + (\frac{2^6 5^{\frac{1}{4}} C_0^3}{\nu^{\frac{3}{4}}}) k \|F_2^n\|^2. \quad (26)$$

Plugging (26) into (25), we see that

$$\|\theta^n\|_{L^4(\mathcal{M})}^2 \leq (1 + \frac{2k}{5})\|\theta^{n-1}\|_{L^4(\mathcal{M})}^2 + (\frac{2^6 5^{\frac{1}{4}} C_0^3}{\nu^{\frac{3}{4}}}) k \|F_2^n\|^2, \quad (27)$$

for $n = 1, 2, \cdots$.

Assuming $k \leq 1$ and applying the discrete Gronwall lemma, we obtain that

$$\begin{aligned}\|\theta^n\|^2_{L^4(\mathcal{M})} &\leq \|\theta_0\|^2_{L^4(\mathcal{M})} \exp(\frac{2nk}{5}) \\ &+ k(1 + \frac{\exp(\frac{2nk}{5}) - 1}{\exp(\frac{2k}{5}) - 1}) \frac{2^6 5^{\frac{1}{4}} C_0^3}{\nu^{\frac{3}{4}}} \|F_2\|^2_\infty \\ &\leq (\|\theta_0\|^2_{L^4(\mathcal{M})} + 2^8 C_0^3 \nu^{-\frac{3}{4}} \|F\|^2_\infty) \exp(\frac{2nk}{5}). \qquad (28)\end{aligned}$$

By Poincare inequality and Sobolev inequality, there exists a constant C_3 such that

$$\|\theta^n\|^2_{L^4(\mathcal{M})} \leq C_3 \|\nabla_3 \theta^n\|^2_{L^2(\mathcal{M})}. \tag{29}$$

Hence, by (29) and (23), we see that

$$\sum_{n=j}^{j+n_2} k\|\theta^n\|^2_{L^4(\mathcal{M})} \leq \sum_{n=j}^{j+n_2} C_3 k \|\nabla_3 \theta^n\|^2_{L^2(\mathcal{M})}$$
$$\leq \frac{2C_3}{\beta\nu}\Big((1+\frac{1}{2}k\nu\lambda)^{-(j-1)}\|u_0\|^2$$
$$+\Big(1-(1+\frac{1}{2}k\nu\lambda)^{-(j-1)}\Big)\frac{4\|F\|^2_\infty}{\nu^2\lambda^2} + \frac{2k(n_2+1)}{\nu\lambda}\|F\|^2_\infty\Big),$$

for any $j, n_2 \in \mathbb{N}$.

Assuming further $k < \frac{2}{\nu\lambda}$, by using the fact $1 + 2x \geq e^x$ for $0 < x < 1$, we see that

$$\sum_{n=j}^{j+n_2} k\|\theta^n\|^2_{L^4(\mathcal{M})} \leq \frac{2C_3}{\beta\nu}\Big(\exp(-\frac{(j-1)k\nu\lambda}{4})\|u_0\|^2 + (\frac{4}{\nu^2\lambda^2} + \frac{2k(n_2+1)}{\nu\lambda})\|F\|^2_\infty\Big).$$

Discrete uniform Gronwall lemma gives

$$\|\theta^n\|^2_{L^4(\mathcal{M})} \leq \Big(\frac{2C_3}{n_2 k\beta\nu}\big[\exp(-\frac{(j-1)k\nu\lambda}{4})\|u_0\|^2 + (\frac{4}{\nu^2\lambda^2} + \frac{2k(n_2+1)}{\nu\lambda})\big]\|F\|^2_\infty + \frac{2^7 C_0^3 (n_2+1)k}{\nu^{\frac{3}{4}}}\|F\|^2_\infty\Big)\exp(\frac{2(n_2+1)k}{5}), \qquad (30)$$

for any positive integer $n \geq j + n_2 + 1$.

Combining (28) and (30) and choosing $j = 1$, we obtain that

$$\|\theta^n\|^2_{L^4(\mathcal{M})} \leq K_\theta, \text{ for } n = 1, 2, 3, \cdots , \tag{31}$$

where

$$K_\theta = K_\theta(n_2, k) = \max\Big\{(\|\theta_0\|^2_{L^4(\mathcal{M})} + 2^8 C_0^3 \nu^{-\frac{3}{4}} \|F\|^2_\infty)\times \\ \exp(\frac{2(n_2+2)k}{5}), \\ \Big(\frac{2C_3}{n_2 k \beta \nu}\big[\|u_0\|^2 + (\frac{4}{\nu^2\lambda^2} + \frac{2k(n_2+1)}{\nu\lambda})\big]\|F\|^2_\infty \\ + \frac{2^7 C_0^3 (n_2+1)k}{\nu^{\frac{3}{4}}}\|F\|^2_\infty\Big)\exp(\frac{2(n_2+1)k}{5})\Big\}$$

Then, by taking the $L^2(\mathcal{M})$ inner product of (1) with $4k|\tilde{\boldsymbol{v}}^n|^2\tilde{\boldsymbol{v}}$, we obtain

$$
\begin{aligned}
\|\tilde{\boldsymbol{v}}^n\|^4_{L^4(\mathcal{M})} - \|\tilde{\boldsymbol{v}}^{n-1}\|^4_{L^4(\mathcal{M})} &+ 2\int_{\mathcal{M}} |\tilde{\boldsymbol{v}}^n|^2|\tilde{\boldsymbol{v}}^n - \tilde{\boldsymbol{v}}^{n-1}|^2 d\mathcal{M} \\
&+ k\nu \int_{\mathcal{M}} \Big(2|\tilde{\boldsymbol{v}}^n|^2|\nabla_3\tilde{\boldsymbol{v}}^n|^2 + |\nabla_3|\tilde{\boldsymbol{v}}|^2|^2\Big) d\mathcal{M} \\
&\leq \Big(\frac{2^{15}3^7C_0^4}{h^3\nu^3} + \frac{6^7C_0^8}{h^4\nu^3}\Big) k\|\boldsymbol{v}^n\|^2\|\nabla_2\boldsymbol{v}^n\|^2\|\tilde{\boldsymbol{v}}^n\|^4_{L^4(\mathcal{M})} \\
&+ \frac{72k(h+1)^2}{\nu}\|\theta^n\|^2_{L^4(\mathcal{M})}\|\tilde{\boldsymbol{v}}^n\|^2_{L^4(\mathcal{M})} \\
&+ \frac{5}{2}\Big(\frac{3}{\nu}\Big)^{\frac{3}{5}} C_0^{\frac{12}{5}} k\|\tilde{F}_1^n\|^{\frac{8}{5}}\|\tilde{\boldsymbol{v}}^n\|^{\frac{12}{5}}_{L^4(\mathcal{M})}.
\end{aligned}
\tag{32}
$$

If $\|\tilde{\boldsymbol{v}}^n\|_{L^4(\mathcal{M})} \neq 0$, dividing (32) by $\|\tilde{\boldsymbol{v}}^n\|^2_{L^4(\mathcal{M})} + \|\tilde{\boldsymbol{v}}^{n-1}\|^2_{L^4(\mathcal{M})}$, we obtain that

$$\begin{aligned}
\|\tilde{\boldsymbol{v}}^n\|^2_{L^4(\mathcal{M})} \leq & \|\tilde{\boldsymbol{v}}^{n-1}\|^2_{L^4(\mathcal{M})} + (\frac{2^{15}3^7 C_0^4}{h^3\nu^3} + \frac{6^7 C_0^8}{h^4\nu^3}) k \|\boldsymbol{v}^n\|^2 \|\nabla_2 \boldsymbol{v}^n\|^2 \times \\
& \|\tilde{\boldsymbol{v}}^n\|^2_{L^4(\mathcal{M})} \\
& + \frac{72k(h+1)^2}{\nu} \|\theta^n\|^2_{L^4(\mathcal{M})} + \frac{5}{2}(\frac{3}{\nu})^{\frac{3}{5}} C_0^{\frac{2}{5}} k \|\tilde{F}_1^n\|^{\frac{8}{5}} \|\tilde{\boldsymbol{v}}^n\|^{\frac{12}{5}}_{L^4(\mathcal{M})} \\
\leq & \Big((\frac{2^{15}3^7 C_0^4}{h^3\nu^3} + \frac{6^7 C_0^8}{h^4\nu^3}) k \|\boldsymbol{v}^n\|^2 \|\nabla_2 \boldsymbol{v}^n\|^2 + \frac{1}{2} k \Big) \|\tilde{\boldsymbol{v}}^n\|^2_{L^4(\mathcal{M})} \\
& + \|\tilde{\boldsymbol{v}}^{n-1}\|^2_{L^4(\mathcal{M})} + \frac{72k(h+1)^2}{\nu} \|\theta^n\|^2_{L^4(\mathcal{M})} \\
& + 2C_0^3 (\frac{3}{\nu})^{\frac{3}{4}} k \|\tilde{F}_1^n\|^2
\end{aligned} \quad (33)$$

By (20), (22) and $k < \frac{2}{\nu\lambda}$, we observe that

$$
\begin{aligned}
k\|\nabla_3 \boldsymbol{v}^n\|^2 &\le \frac{2}{\nu}\Big((1+\frac{1}{2}k\nu\lambda)^{-(n-1)}\|u_0\|^2 + \frac{8}{\nu^3\lambda^2}\|F\|_\infty^2 + \frac{2k}{\nu\lambda}\|F\|_\infty^2\Big) \\
&\le \frac{4}{\nu}K_1.
\end{aligned}
$$

By choosing $\|u_0\|^2$, $\|F\|_\infty$ and k small enough (to be determined later), we have

$$
\begin{aligned}
(\frac{2^{15}3^7C_0^4}{h^3\nu^3}+\frac{6^7C_0^8}{h^4\nu^3})k\|\boldsymbol{v}^n\|^2\|\nabla_2\boldsymbol{v}^n\|^2 + \frac{1}{2}k \le (\frac{2}{\nu})(\frac{2^{15}3^7C_0^4}{h^3\nu^3}+\frac{6^7C_0^8}{h^4\nu^3})K_1^2 \\
+\frac{1}{2}k < \frac{1}{2}.
\end{aligned}
\tag{34}
$$

We then derive from (33) that

$$\|\tilde{\boldsymbol{v}}^n\|^2_{L^4(\mathcal{M})} \le 2\Big(\|\tilde{\boldsymbol{v}}^{n-1}\|^2_{L^4(\mathcal{M})} + \frac{72k(h+1)^2}{\nu}\|\theta^n\|^2_{L^4(\mathcal{M})} + 2C_0^3(\frac{3}{\nu})^{\frac{3}{4}}k\|\tilde{F}_1^n\|^2\Big). \tag{35}$$

Note that

$$\begin{cases} \|\tilde{F}_1^n\| \le 2\|F\|_\infty, \\ \|\boldsymbol{v}^n\|^2 \le K_1, \\ \|\theta^n\|_{L^4(\mathcal{M})} \le K_\theta, \end{cases} \tag{36}$$

for all $n \in \mathbb{N}$, we get

$$\begin{aligned} \|\tilde{\boldsymbol{v}}^n\|_{L^4(\mathcal{M})}^2 \le & \Big(1 + 2\Big(\big(\frac{2^{15}3^7C_0^4}{h^3\nu^3} + \frac{6^7C_0^8}{h^4\nu^3}\big)K_1k\|\nabla_2\boldsymbol{v}^n\|^2 + \frac{1}{2}k\Big)\Big) \\ & \times \|\tilde{\boldsymbol{v}}^{n-1}\|_{L^4(\mathcal{M})}^2 \\ & + 2\Big(\frac{72k(h+1)^2}{\nu}K_\theta + 8C_0^3\big(\frac{3}{\nu}\big)^{\frac{3}{4}}k\|F\|_\infty^2\Big). \end{aligned} \tag{37}$$

Thank you very much for your attention!!