Algebraic Voting Theory

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Algebraic Combinatorixx

Rank Aggregation

Big problem: Given a set of voters with preferences on candidates, find a “winner".

Input	Map	Output

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Approval Sets		
Partial Orders		

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Full Rankings		Winning Candidate
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Partial Orders		Committees
		Partial Orders

Rank Aggregation

Big problem: Given a set of voters with preferences on candidates, find a “winner".

Input	Map	Output
Full Rankings	Positional scoring	Winning Candidate
Approval Sets	Approval voting	Full Rankings
Partial Orders	Condorcet method	Committees
	Kemeny Rule	Partial Orders

Kenneth Arrow

No perfect system. All aggregation techniques require some compromise

Don Saari

Rather than voter preference, an election outcome can reflect the choice of an election method.
"For a price... I will come to your group before your next election. You tell me who you want to win..."

Example: Fruit Selection

Suppose we want to know the most preferred fruit out of apple, banana and cranberry.

# of voters	Preference ordering
3	Apple, Banana, Cranberry
2	Apple, Cranberry, Banana
0	Banana, Apple, Cranberry
2	Banana, Cranberry, Apple
0	Cranberry, Apple, Banana
4	Cranberry, Banana, Apple

Example: Fruit Selection

Votes from the election are aggregated in a **profile vector**.

$$\vec{p} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} \begin{matrix} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{matrix}$$

A positional weight function assigns values to the candidates.

Example: Fruit Selection

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$$\vec{p} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} \begin{matrix} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{matrix} \xrightarrow{\text{Plurality}} \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$

$$\vec{w} = (1, 0, 0)$$

Example: Fruit Selection

Votes from the election are aggregated in a **profile vector**.

$$\vec{p} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} \begin{matrix} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{matrix} \xrightarrow{\text{Borda}} \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$

$$\vec{w} = (2, 1, 0)$$

Voting Systems as Linear Transformations

Positional voting methods can be represented as linear transformations:

$$T_{\vec{w}} : \text{profile space} \to \text{results space}$$

Example

$$T_{(2,1,0)}\vec{p} = \begin{pmatrix} \overset{ABC}{2} & \overset{ACB}{2} & \overset{BAC}{1} & \overset{BCA}{0} & \overset{CAB}{1} & \overset{CBA}{0} \\ 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}$$

A representation theory approach

Daugherty, Eustis, Minton and Orrison (2009)

View profile and results spaces as permutation modules and positional maps as module homomorphisms.

Inputs: Tabloids of a given shape λ.

B	D	
C		
A	E	F

Definition

M^λ is the vector space over $\mathbb{Q}$ generated by the λ-tabloids.

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Example, $n = 3$

$$\text{Full rankings} \xrightarrow{\text{Borda}} \text{Winning Candidate}$$

$$M^{(1,1,1)} \xrightarrow{T_{(2,1,0)}} M^{(1,2)}$$

$$3\ \begin{array}{|c|}\hline A\\\hline B\\\hline C\\\hline\end{array} + 2\ \begin{array}{|c|}\hline A\\\hline C\\\hline B\\\hline\end{array} + \ldots + 4\ \begin{array}{|c|}\hline C\\\hline B\\\hline A\\\hline\end{array} \mapsto 10\ \begin{array}{|c|c}\hline A & \multicolumn{1}{c}{}\\\hline B & \multicolumn{1}{c|}{C}\\\hline\end{array} + 11\ \begin{array}{|c|c}\hline B & \multicolumn{1}{c}{}\\\hline A & \multicolumn{1}{c|}{C}\\\hline\end{array} + 12\ \begin{array}{|c|c}\hline C & \multicolumn{1}{c}{}\\\hline A & \multicolumn{1}{c|}{B}\\\hline\end{array}$$

Permutation modules

There is a natural action of S_n on tabloids. For $\sigma = (A, B, C)$,

$$\sigma \begin{array}{cc} \hline A & B \\ \hline C & \\ \hline D & \\ \hline \end{array} = \begin{array}{cc} \hline \sigma(A) & \sigma(B) \\ \hline \sigma(C) & \\ \hline \sigma(D) & \\ \hline \end{array} = \begin{array}{cc} \hline B & C \\ \hline A & \\ \hline D & \\ \hline \end{array}$$

Extending this action of S_n to M^λ makes M^λ a QS_n-module

Schur's Lemma

M^λ can be decomposed into irreducible submodules, indexed by partitions of n.

$$M^{(1,1,1)} \cong S^{(3)} \oplus 2S^{(2,1)} \oplus S^{(1,1,1)}$$
$$M^{(1,2)} \cong S^{(3)} \oplus S^{(2,1)}$$

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Effective Space

The kernel of T_w contains $S^{(1,1,1)}$ and at least one copy of $S^{(2,1)}$.

$$S^{(1,1,1)} \cong \left\langle \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle \begin{matrix} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{matrix}$$

Profile space = $\text{Ker}(T_w) \oplus \text{Ker}(T_w)^{\perp} = \text{Ker}(T_w) \oplus E(T_w)$

$E(T_w) \cong Im(T_w)$ is the effective space

Theorem (Daugherty, Eustis, Minton, Orrison)

If $w \not\sim w'$ then $E(T_w) \cap E(T_{w'}) = \{0\}$

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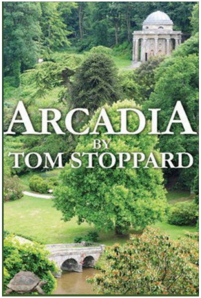
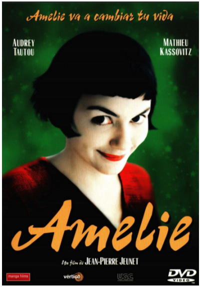
Committee Voting

As a second example, consider choosing a pair of films to show at your conference.



Best Depiction of a Mathematician



Bechdel Test Honorable Mention

Committee voting: Stephen Lee (2010)

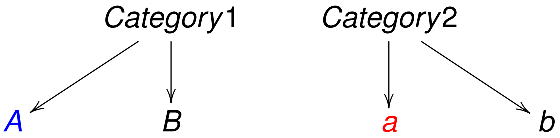
For n departments, with m candidates in each department,

- Voters rank all possible committees: profile space is $(m^n)!$-dim.
- Results space is (m^n)-dim v.s. generated by committees.
- There is a natural action of $S_m[S_n]$ on the committees, where

$$S_m[S_n] = \{(\sigma_1, \ldots, \sigma_n; \pi) : \sigma_i \in S_m, \pi \in S_n\}.$$

Example: Selecting award winners with $S_2[S_2]$

For example, consider
$\varphi = ((12), e; (12))$ acting on nominee set $\{A, a\}$



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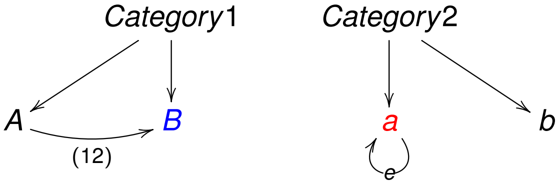
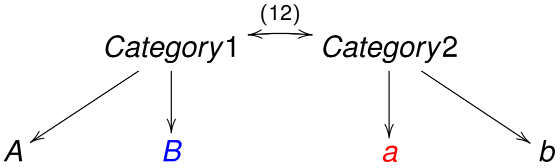


$\varphi(\{A, a\}) \Rightarrow (\{B, a\})$

Example: Selecting award winners with $S_2[S_2]$

$\varphi = ((12), e; (12))$ acting on committee $W = \{a_1, b_1\}$



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Selecting committees with $S_2[S_2]$

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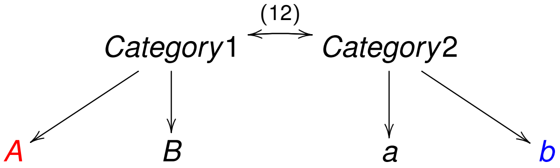

$\varphi(\{A, a\}) \Rightarrow (\{B, a\}) \Rightarrow (\{A, b\})$

$\mathbb{Q}S_m[S_n]$-modules

The action of $S_m[S_n]$ on the profile P and results R spaces makes them $\mathbb{Q}(S_m[S_n])$-modules.
$T_w : P \to R$ is a $\mathbb{Q}S_m[S_n]$-module homomorphism.

Irreducible submodules of a $\mathbb{Q}S_m[S_n]$-module are indexed by tuples of partitions which add up to n.

$$S_3[S_5]$$

$$S^{((2),(1),(1,1))}, \qquad S^{((2,2),\emptyset,(1))}$$

Example: The $S_2[S_2]$ case

$$S^{(\square\square,\emptyset)} = \left\langle \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \right\rangle \qquad S^{(\emptyset,\square\square)} = \left\langle \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} \right\rangle \begin{pmatrix}Aa\\Ab\\Ba\\Bb\end{pmatrix}$$

$$S^{(\substack{\square\\\square},\emptyset)} = S^{(\emptyset,\substack{\square\\\square})} = \vec{0} \qquad S^{(\square,\square)} = \left\langle \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \right\rangle$$

Let R_2 be the $\mathbb{Q}S_2[S_2]$-module results space spanned by the committees consisting of exactly 1 member from each of 2 departments.

$$R_2 \cong S^{(\square\square,\emptyset)} \oplus S^{(\square,\square)} \oplus S^{(\emptyset,\square\square)}$$

Decompositions of R_n

$$R_2 \cong S^{(\square\square,\emptyset)} \oplus S^{(\square,\square)} \oplus S^{(\emptyset,\square\square)}$$
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Conjecture (Lee, 2010 Thesis)

For $S_2[S_n]$ with $n \geq 2$, the results space decomposes into exactly $\bigoplus_\lambda S^\lambda$, the direct sum of irreducible submodules indexed by double trivial partitions $\lambda = (\lambda_1, \lambda_2)$ (the "flat" partitions).

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Theorem (Matt Davis, 2010)

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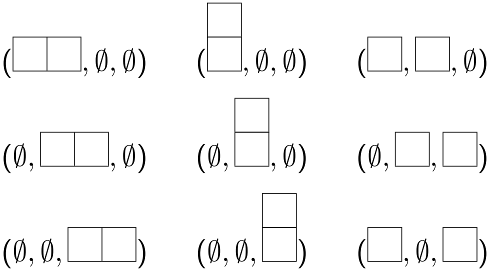
Decomposing a $\mathbb{Q}S_3[S_2]$-module

Conjecture (Calaway, Csapo, Samelson, 2015)

For $S_m[S_n]$ with $m, n \geq 2$, the results space decomposes into a direct sum composed only of irreducible submodules indexed by h-tuple trivial partitions (the "flat" partitions).



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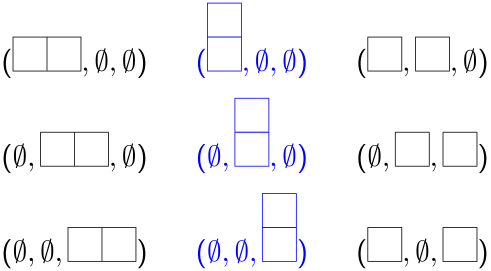

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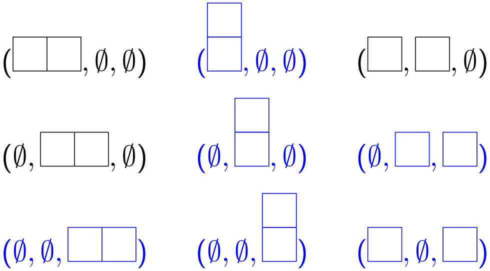
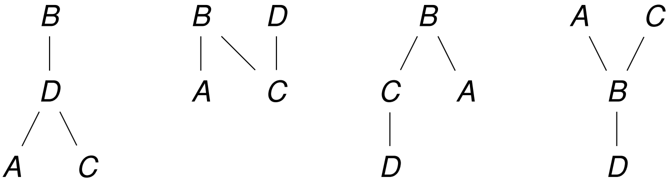


Voting on Posets

Suppose there is one “correct” poset and the votes are noisy approximations of it

Input:



Output: Most likely correct poset

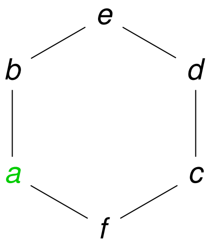
Borda Count for Posets

Cullinan, Hsiao, and Polett (2014) define a Borda extension for posets.

Definition

The Borda score a candidate a receives for a poset P is

$$b(P, a) = 2\ \mathrm{down}(a) + 1\ \mathrm{incomp}(a)$$



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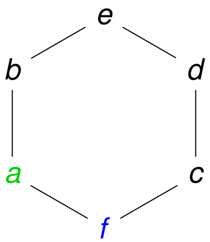

$b(P, a) = 2\ \mathrm{down}(a) + 1\ \mathrm{incomp}(a) = 2(1) +$

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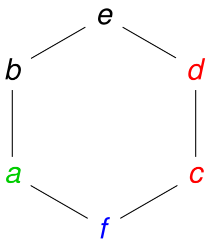

$b(P, a) = 2\ \mathrm{down}(a) + 1\ \mathrm{incomp}(a) = 2(1) + 1(2) = 4$

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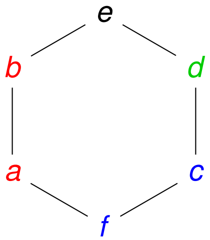

$B(d) = 2\ \text{down}(d) + 1\ \text{incomp}(d) = 2(2) + 1(2) = 6$

Scoring Rankings: SRSF

Let A be the set of candidates and V our voter profile.
For u and v full rankings, and t a positional scoring function,

$$s(v,u) = \sum_{i=1}^{m} (m-i)t(v,u(i)) = \sum_{a \in A} b(u,a)t(v,a)$$

Then we can compare each full ranking to our profile

$$S(u) = \sum_{v \in V} s(v,u)$$

Theorem (Conitzer, Rognlie, Xia '09)

A (neutral) voting function is a maximum likelihood estimator if and only if it is an SRSF.

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S' is *consistent* with t:
If $t(a) > t(b)$, then $a \geq b$ in all winning posets.

However, linear extensions of winning posets are also winning.

Can we extend in another way to compare the poset structures too?

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