Dyck Paths and Positroids from Unit Interval Orders

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Algebraic Combinatorixx 2

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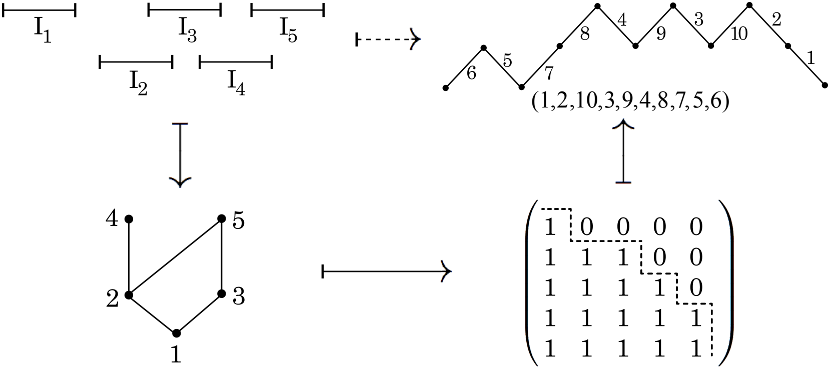
A Pictorial Guide



Outline

Unit Interval Orders

Unit Interval Orders

Definition

A poset P is a *unit interval order* if there exists a bijective map $i \mapsto [q_i, q_i + 1]$ from P to $S = \{[q_i, q_i + 1] \mid 1 \leq i \leq n,\ q_i \in \mathbb{R}\}$ such that for distinct $i, j \in P$, $i <_P j$ if and only if $q_i + 1 < q_j$. We then say that S is an *interval representation* of P.

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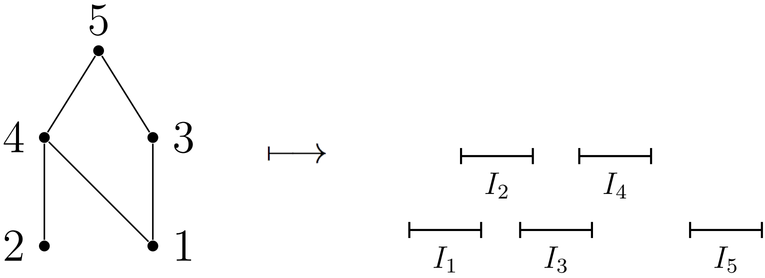

Subposets of Unit Interval Orders

- A subset Q is an *induced* subposet of P if there is an injective map $f \colon Q \to P$ such that $r <_Q s$ if and only if $f(r) <_P f(s)$.
- P is a Q-*free* poset if P does not contain any induced subposet isomorphic to Q.

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Theorem (Scott–Suppes)

A poset is a unit interval order if and only if it is simultaneously $(\mathbf{3}+\mathbf{1})$*-free and* $(\mathbf{2}+\mathbf{2})$*-free.*

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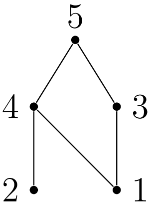

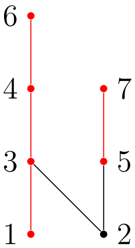

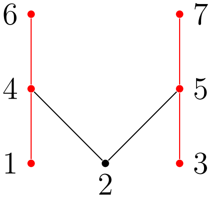

Natural and Altitude Preserving Labelings

Let P be a poset on $[n]$.

- P is *naturally labeled* if $i <_P j$ implies that $i < j$ as integers.
- A labeling on P is *altitude preserving* if $\alpha(i) < \alpha(j)$ implies $i < j$ (as integers), where $\alpha(i) = |\Lambda_i| - |\mathrm{V}_i|$ is called the *altitude* of i.

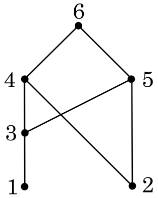

Figure: A poset with an altitude preserving labeling on [6].

Positroids

Matroids

Definition (Matroid)

Let E be a finite set, and let $\mathcal{B}$ be a nonempty collection of subsets, called *bases*, of E. The pair $M = (E, \mathcal{B})$ is a *matroid* if they satisfy the ***Basis Exchange Axiom***:

- for all $A, B \in \mathcal{B}$ and $a \in A \setminus B$, there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

Example: Given the bases

$$\mathcal{B} = \big\{\{2,4,6\}, \{2,5,6\}\big\},$$

then the pair $M = ([6], \mathcal{B})$ a matroid.

Matroid Example

Consider the 3×6 real matrix

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

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- Label the columns 1 through 6 and notice X has rank 3.

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- Label the columns 1 through 6 and notice X has rank 3.
- Then $B \in \mathcal{B}$ is a set of 3 columns that span $\mathbb{R}^3$.
- The matroid represented by X is $M = ([6], \mathcal{B})$ with bases

$$\mathcal{B} = \big\{\{2,4,6\},\{2,5,6\}\big\}.$$

Positroids

Definition: A matroid $([n], \mathcal{B})$ of rank d is *representable* if there is $X \in M_{d\times n}(\mathbb{R})$ with columns $X_1, \ldots, X_n$ such that $B \subseteq [n]$ belongs to $\mathcal{B}$ iff $\{X_i \mid i \in B\}$ is a basis for $\mathbb{R}^d$.

Definition (Positroid)

A *positroid* on $[n]$ of rank d is a matroid that can be represented by a matrix in $\mathrm{Mat}^{+}_{d,n}$.

Notation: Let $\mathrm{Mat}^{\geq 0}_{d,n}$ denote the set of all full rank $d \times n$ real matrices with nonnegative maximal minors.

Example: Recall the 3×6 real matrix

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

- All maximal minors are nonnegative, thus $X \in \mathrm{Mat}^{+}_{3,6}$.
- The matroid $M = ([6], \mathcal{B})$ represented by X is a positroid.

Dyck Matrices

Definition (Dyck Matrix)

A binary square matrix is said to be a *Dyck matrix* if its zero entries are above the main diagonal and its one entries are separated from its zero entries by a Dyck path supported on the main diagonal. We let $\mathcal{D}_n$ denote the set of Dyck matrices of size n.

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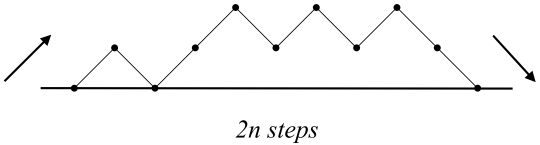
A *Dyck path*



Dyck Matrices

Example: A 6×6 Dyck matrix and its Dyck path:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Observations:

- Every Dyck matrix is totally nonnegative.
- $|\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.

Note: A square matrix is *totally nonnegative* if all its minors are ≥ 0.

Antiadjacency Matrices of Labeled Posets

Definition (Antiadjacency Matrix)

If P is a poset $[n]$, then the *antiadjacency matrix* of P is the $n \times n$ binary matrix $A = (a_{i,j})$ with $a_{i,j} = 0$ iff $i \neq j$ and $i <_P j$.

Proposition (Skandera–Reed)

An n-labeled unit interval order has an altitude preserving labeling if and only if its antiadjacency matrix is a Dyck matrix.

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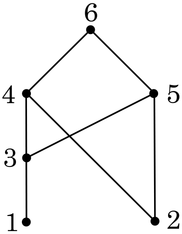

$$\longmapsto \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Postnikov's Map

Lemma (Postnikov)

For an $n \times n$ real matrix $A = (a_{i,j})$, consider the $n \times 2n$ matrix $B = \phi(A)$, where

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n} \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \overset{\phi}{\mapsto} \begin{pmatrix} 1 & \dots & 0 & 0 & \pm a_{n,1} & \dots & \pm a_{n,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & -a_{2,1} & \dots & -a_{2,n} \\ 0 & \dots & 0 & 1 & a_{1,1} & \dots & a_{1,n} \end{pmatrix}.$$

Under this correspondence, $\Delta_{I,J}(A) = \Delta_{(n+1-[n]\setminus I)\cup(n+J)}(B)$ for all $I, J \subseteq [n]$ satisfying $|I| = |J|$ (here $\Delta_{I,J}(A)$ is the minor of A determined by the rows I and columns J, and $\Delta_K(B)$ is the maximal minor of B determined by columns K).

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This allows us to associate a positroid to each Dyck matrix

Postnikov's Map

Example: By Lemma, the Dyck matrix A produces the postroid represented by $\phi(A)$:

$$\underset{A}{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \mapsto \underset{\phi(A)}{\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}}$$

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where the minors of A correspond with maximal minors of $\phi(A)$:

For row index set $I = \{1, 2\}$ and column index set $J = \{2, 3\}$, we have $\Delta_{I,J}(A) = \Delta_{\{1,5,6\}}(\phi(A))$.

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Thus, every Dyck matrix produces a positroid.

Unit Interval Positroids

Definition (Unit Interval Positroid)

For $D \in \mathcal{D}_n$, the positroid on $[2n]$ represented by $\phi(D)$ is called a *unit interval positroid*. Let $\mathcal{P}_n$ denote the set of all unit interval positroids on $[2n]$.

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For every n, the following sequence of maps is bijective:

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Unit Interval Orders $\leftrightarrow$ Dyck matrices $\leftrightarrow$ Unit Interval Positroids

Corollary: There are $\frac{1}{n+1}\binom{2n}{n}$ unit interval positroids on $[2n]$.

Decorated Permutations and Unit Interval Positroids

Decorated Permutations

Definition (Decorated Permutation)

A *decorated permutation* of $[n]$ is an element $\pi \in S_n$ whose fixed points j are marked either "clockwise"(denoted by $\pi(j) = \underline{j}$) or "counterclockwise" (denoted by $\pi(j) = \overline{j}$).

Example:

$$26\bar{3}145 = (12654)(\bar{3})$$

Example of a Decorated Permutation

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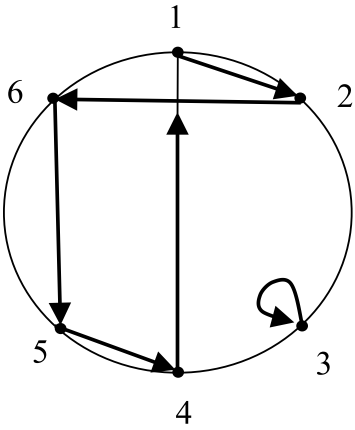

Decorated Permutations of Unit Interval Positroids

Theorem (C-G)

Number the n vertical steps of the Dyck path of $D \in \mathcal{D}_n$ from bottom to top with $1, \ldots, n$ and the n horizontal steps from left to right with $n+1, \ldots, 2n$. Then the decorated permutation of the unit interval positroid induced by D is obtained by reading the Dyck path of D in the northwest direction.

Decorated Permutation from Dyck Matrix

Example: The decorated permutation π associated to the positroid represented by the 5×5 Dyck matrix D

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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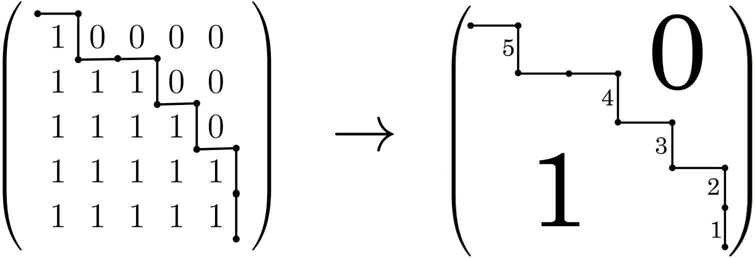

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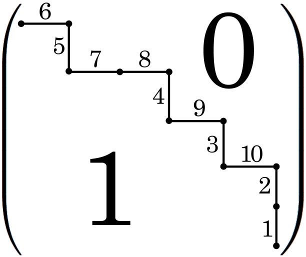

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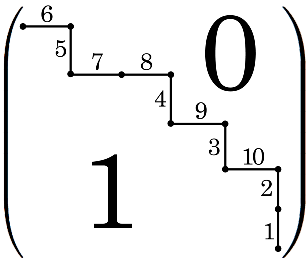

can be read from the Dyck path of D, obtaining

$$\pi = (1, 2, 10, 3, 9, 4, 8, 7, 5, 6).$$

Decorated Permutations of Unit Interval Positroids

Theorem (C-G)

Decorated permutations associated to unit interval positroids on $[2n]$ *are* $2n$*-cycles* $(1\ j_1\ \dots\ j_{2n-1})$ *satisfying the following two conditions:*

1. *in the sequence* $(1, j_1, \dots, j_{2n-1})$ *the elements* $1, \dots, n$ *appear in increasing order while the elements* $n+1, \dots, 2n$ *appear in decreasing order;*
2. *for every* $1 \leq k \leq 2n-1$*, the set* $\{1, j_1, \dots, j_k\}$ *contains at least as many elements of the set* $\{1, \dots, n\}$ *as elements of the set* $\{n+1, \dots, 2n\}$*.*

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The decorated permutation of a unit interval positroid is a Dyck path.

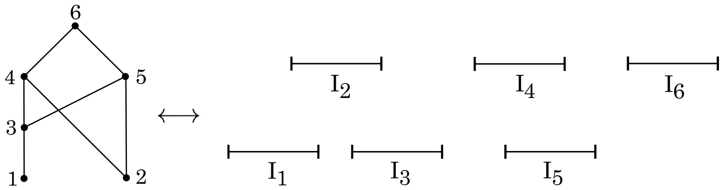
Decorated Permutations and Interval Representations

Canonical Interval Representation

Proposition (C-G)

Let P be a unit interval order on $[n]$. Then the labeling of P preserves altitude if and only if there exists an interval representation $\{[q_i, q_i + 1] \mid 1 \leq i \leq n\}$ of P such that $q_1 < \cdots < q_n$.

Example:

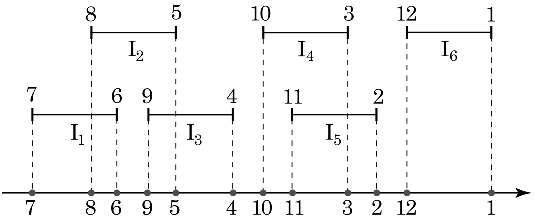


Decorated Permutation Read from Canonical Interval Representation

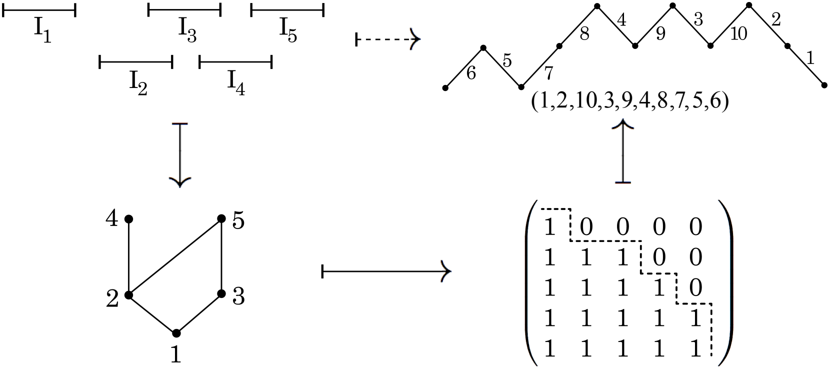
Theorem (C-G)

Labeling the left and right endpoints of the intervals $[q_i, q_i + 1]$ *by* $n + i$ *and* $n + 1 - i$*, respectively, we obtain the decorated permutation of the positroid induced by* P *by reading the label set* $\{1, \dots, 2n\}$ *from the real line from right to left.*

Example: The decorated permutation $(1, 12, 2, 3, 11, 10, 4, 5, 9, 6, 8, 7)$ is obtained by reading the labels from right to left.



Intervals to Positroids and back



Thank you!

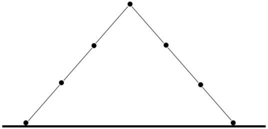
email: a.chavez@berkeley.edu

Reference: A. Chavez and F. Gotti. *Dyck Paths and Positroids from Unit Interval Orders*. https://arxiv.org/abs/1611.09279.

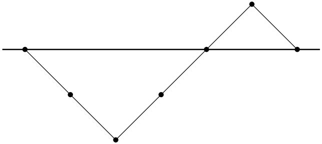
Acknowledgments:
Thank you to Dr. Federico Ardila and Dr. Lauren Williams for their ongoing support.

Special thanks to Alejandro Morales for suggesting the question that motivated this project.

What's in a name?



$(1, 2, 3)$



$(3, 4, 5)$