Recent Development on Dynamic Computer Experiments

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Outline

- Large-scale dynamic computer experiments
- Proposed SVD-based Gaussian process models
- Inverse problem
- Concluding remarks

Dynamic Computer Simulators

Computer simulators with time-varying outputs

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Computer simulators with time-varying outputs

- Climate change model (Bhattacharya, 2007)
- Rainfall-runoff model (Conti et al., 2009)
- Marrow stem cell kinetics model (Morris, 2012; Jones et al., 1991)
- Vehicle suspension system (Bayrri et al., 2007)
- Hydrological saturated path model (logSPMs) (Liu and West, 2007)
- Sheffield Dynamic Global Vegetation Model (Conti and O'Hagan, 2010)
- The Lyon-Fedder-Mobary (LFM) model (Pratola et al., 2013)
- Volcanic pyroclastic flow model (Gu and Berger, 2016)

Dynamic Computer Simulators

- Systems evolving over time

$$Y_t = f(z_t, Y_{t-1}), t = 1, \ldots, T \tag{1}$$

where Y_t and Y_{t-1} represent the state vector at time-step t and $t-1$, respectively, $\mathbf{z}_t = (\mathbf{x}, \mathbf{w}_t)$ denotes the time-independent input $\mathbf{x}$ and the forcing inputs $\mathbf{w}_t$ at time-step t

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- Systems generating time series outputs
  - the training data inputs $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$
  - at time t, the outputs $\mathbf{y}_t = (y(\mathbf{x}_1, t), \ldots, y(\mathbf{x}_N, t))^T$

Dynamic Computer Simulators

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- Time-varying autoregression (TVAR) models with GP residuals (Liu and West, 2007)

Large-scale Computer Experiments

Computer Experiments with massive outputs

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- Mutli-step interpolator (Haaland and Qian, 2011)
- Fixed rank kriging (Cressie and Johannesson, 2008)
- Compactly supported covariance (Kaufman et al., 2011)
- Local Gaussian process (Gramacy and Apley, 2015; Gramacy et al., 2015; Gramacy and Haaland, 2016)

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- Emery (2009) built a local neighborhood by sequentially including data that make the Kriging variance decrease more.
- Gramacy and Apley (2015) uses a sequential greedy algorithm and an optimality criterion for finding a non-trivial local neighbourhood set.

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- It is not trivial to extend the exiting methods for large-scale computer experiments to accommodate the times series outputs
- Thus, it calls for a new method for emulating large-scale dynamic computer simulators

The Proposed Approach

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- An optimality criterion to search for next point

Model Formulation

- For N training points, let $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_N]^T$ be the $N \times q$ input matrix, and $\mathbf{Y} = [\mathbf{y}(\mathbf{x}_1), \ldots, \mathbf{y}(\mathbf{x}_N)]$ be the $L \times N$ matrix of time series response

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- The SVD on $\mathbf{Y}$ gives

$$\mathbf{Y} = \mathbf{U}\mathbf{D}\mathbf{V}^T = \sum_{i=1}^{k} d_i \mathbf{u}_i \mathbf{v}_i^T \tag{3}$$

where

- $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_k]$ is an $L \times k$ column orthogonal matrix of left singular vectors, $k = \min\{N, L\}$
- $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_k)$ is a $k \times k$ diagonal matrix of singular values sorted in decreasing order
- $\mathbf{V}$ is an $N \times k$ column-orthogonal matrix of right singular vectors

Model Formulation

The SVD-based GP model assumes that

$$\mathbf{y}(\mathbf{x}) = \sum_{i=1}^{p} c_i(\mathbf{x})\mathbf{b}_i + \varepsilon \tag{4}$$

where $\mathbf{b}_i = d_i\mathbf{u}_i \in \mathcal{R}^L$, $c_i(\mathbf{x}) \sim \mathrm{GP}(0, \sigma_i^2 K_i(\cdot,\cdot;\theta_i))$, for $i = 1,\ldots,p$, and $\varepsilon \sim N(0, \sigma^2\mathbf{I}_L)$. The number of significant singular values, p in (4), is determined empirically by the cumulative percentage criterion

$$p = \min\left\{ m : \frac{\sum_{i=1}^{m} d_i}{\sum_{i=1}^{k} d_i} > \gamma \right\},$$

where γ is a prespecified threshold of explained variation ($\gamma \in (0.9, 0.98)$)

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- The formulation reduces the problem of building an emulator that maps from $(0,1)^q$ to $\mathcal{R}^L$ to building p independent, univariate GP models for each c_i
- Higdon et al. (2008) employ a fully Bayesian analysis via Markov chain Monte Carlo (MCMC), which is computationally intensive in our applications

Empirical Bayesian Inference

- The parameters
  - σ^2: the error variance
  - σ_i^2: the process variance, for $i = 1, \ldots, p$
  - $\theta_i = (\theta_{i1}, \ldots, \theta_{iq})$: the correlation parameters in Gaussian correlation function

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  - $\theta_i = (\theta_{i1}, \ldots, \theta_{iq})$: the correlation parameters in Gaussian correlation function
- The priors:
  - $[\sigma^2] \sim \text{IG}\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$
  - $[\sigma_i^2] \sim \text{IG}\left(\frac{\alpha_i}{2}, \frac{\beta_i}{2}\right), i = 1, \ldots, p$
  - $[\frac{1}{\theta_{ij}}] \sim \textit{Gamma}(3/2, a)$, for $i = 1, \ldots, p$ and $j = 1, \ldots, q$, where the scale parameter a is chosen such that maximum squared distance among any two points of the design matrix lies at the position of 95% quantile (Gramacy, 2015)

Empirical Bayesian Inference

The approximate predictive distribution of $\mathbf{y}(\mathbf{x}_0|\mathbf{Y})$:

$$\pi(\mathbf{y}(\mathbf{x}_0)|\mathbf{Y}) \approx \pi(\mathbf{y}(\mathbf{x}_0)|\mathbf{V}^*, \hat{\sigma}^2, \hat{\Theta}) \approx \mathcal{N}\big(\mathbf{B}\hat{\mathbf{c}}(\mathbf{x}_0|\mathbf{V}^*, \hat{\Theta}), \mathbf{B}\Lambda(\mathbf{V}^*, \hat{\Theta})\mathbf{B}^T + \hat{\sigma}^2\mathbf{I}_L\big). \tag{5}$$

where

- $\hat{\mathbf{c}}(\mathbf{x}_0|\mathbf{V}^*, \hat{\Theta}) = [\hat{c}_1(\mathbf{x}_0|\mathbf{v}_1, \hat{\theta}_1), \ldots, \hat{c}_p(\mathbf{x}_0|\mathbf{v}_p, \hat{\theta}_p)]^T$
- $\Lambda(\mathbf{V}^*, \hat{\Theta}) = \mathrm{diag}\big(\hat{\sigma}_1^2(\mathbf{x}_0|\mathbf{v}_1, \hat{\theta}_1), \ldots, \hat{\sigma}_p^2(\mathbf{x}_0|\mathbf{v}_p, \hat{\theta}_p)\big)$.

- $[\sigma^2|Y]$ follows the inverse Gamma distribution $\mathrm{IG}((NL+\alpha)/2, (r^T r+\beta)/2)$, and

$$\hat{\sigma}^2 = \underset{\sigma^2}{\operatorname{argmax}}\, \pi(\sigma^2|Y) = \frac{1}{NL+\alpha+2}\left(r^T r+\beta\right). \tag{6}$$

- $\hat{\theta}_i = \underset{\theta_i}{\operatorname{argmax}}\, \pi(\theta_i|\mathbf{v}_i), i = 1,\ldots,p.$

$$\pi(\theta_i|\mathbf{v}_i) \propto |K_i|^{-\frac{1}{2}} \left(\frac{\beta_i+\psi_i}{2}\right)^{-(\alpha_i+N)/2} \pi(\theta_i), \tag{7}$$

Empirical Bayesian Inference

$$\pi(\mathbf{y}(\mathbf{x}_0)|\mathbf{V}^*, \Theta, \sigma^2) \approx \mathcal{N}\left(\mathbf{B}\hat{\mathbf{c}}(\mathbf{x}_0|\mathbf{V}^*, \Theta), \mathbf{B}\Lambda(\mathbf{V}^*, \Theta)\mathbf{B}^T + \sigma^2\mathbf{I}_L\right), \quad (8)$$

To compute (8), we make use of

- the distribution of $[\mathbf{y}(\mathbf{x}_0)|\mathbf{V}^*, \Theta, \sigma^2, c_1(\mathbf{x}_0), \ldots, c_p(\mathbf{x}_0)]$
- the distribution of $[c_i(\mathbf{x}_0)|\mathbf{v}_i, \theta_i]$

Lemma

(Gelman et al., 2014) Suppose $[\boldsymbol{y}|\beta,\sigma^2] \sim \mathcal{N}(\boldsymbol{X}\beta,\sigma^2\boldsymbol{I}_n)$ *and* $[\beta] \sim \mathcal{N}(\boldsymbol{b},V)$*, where* $\boldsymbol{y} \in \mathbb{R}^n$*,* $\beta,\boldsymbol{b} \in \mathbb{R}^m$*,* $\boldsymbol{X}$ *is an* $n \times m$ *matrix, and* V *is an* $m \times m$ *positive definite covariance matrix. Then,* $[\boldsymbol{y}|\sigma^2] \sim \mathcal{N}(\boldsymbol{X}\boldsymbol{b},\boldsymbol{X}V\boldsymbol{X}^T+\sigma^2\boldsymbol{I}_n)$.

- given $[\mathbf{y}(\mathbf{x}_0)|\mathbf{V}^*,\Theta,\sigma^2,c_1(\mathbf{x}_0),\ldots,c_p(\mathbf{x}_0)]$
- given $[c_i(\mathbf{x}_0)|\mathbf{v}_i,\theta_i]$
- derive $[\mathbf{y}(\mathbf{x}_0)|\mathbf{V}^*,\Theta,\sigma^2]$

Empirical Bayesian Inference

The conditional distribution of $c_i(\mathbf{x}_0)$ given $(\mathbf{v}_i, \theta_i)$ are independent non-central t with $N + \alpha_i$ degrees of freedom, for $i = 1, \ldots, p$, i.e.,

$$[c_i(x_0)|\mathbf{v}_i, \theta_i] \sim t_{N+\alpha_i}\left(\hat{c}_i(\mathbf{x}_0|\mathbf{v}_i, \theta_i), \hat{\sigma}_i^2(\mathbf{x}_0|\mathbf{v}_i, \theta_i)\right), \tag{9}$$

with the location parameter

$$\hat{c}_i(\mathbf{x}_0|\mathbf{v}_i, \theta_i) = k_i^T(\mathbf{x}_0)K_i^{-1}\mathbf{v}_i, \tag{10}$$

with $k_i(\mathbf{x}_0) = [K(\mathbf{x}_0, \mathbf{x}_1; \theta_i), \ldots, K(\mathbf{x}_0, \mathbf{x}_N; \theta_i)]^T$, and the scale parameter

$$\hat{\sigma}_i^2(\mathbf{x}_0|\mathbf{v}_i, \theta_i) = \frac{(\beta_i + \psi_i)\left(1 - k_i^T(\mathbf{x}_0)K_i^{-1}k_i(\mathbf{x}_0)\right)}{\alpha_i + N}. \tag{11}$$

- The posterior distribution of $c_i(\mathbf{x}_0)|\mathbf{v}_i, \theta_i$

$$\pi\left(c_i(\mathbf{x}_0)|\mathbf{v}_i, \theta_i\right) \approx \mathcal{N}\left(\hat{c}_i(\mathbf{x}_0|\mathbf{v}_i, \theta_i), \hat{\sigma}_i^2(\mathbf{x}_0|\mathbf{v}_i, \theta_i)\right). \tag{12}$$

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- The local SVD-based GP model uses n ($\ll N$) points for approximating the predicted response at an arbitrary $\mathbf{x}_0$
- The *naive local SVD-based GP model* uses n nearest neighbour points based on the Euclidean distance
- The proposed *greedy local SVD-based GP model* sequentially finds the n neighbour points with the aim of reducing the prediction error

Greedy Local SVD-based GP Model

- Let k denote the current number of points in the neighbourhood set, $\mathbf{X}^{(k)}$ and $\mathbf{X} \backslash \mathbf{X}^{(k)}$ be the sets of selected and unselected (remaining) training points, respectively, and $\hat{\Theta}^{(k)} = \{\hat{\theta}_1^{(k)}, \ldots, \hat{\theta}_p^{(k)}\}$ be the estimated correlation parameters.

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- The next follow-up point is chosen as

$$x_{k+1}^* = \underset{x \in X \backslash \mathbf{x}^{(k)}}{\operatorname{argmin}} \; J(\mathbf{x}_0, \mathbf{x}),$$

J-Criterion

$$J(\mathbf{x}_0, \mathbf{x}) = \mathrm{E}\left\{ \mathrm{E}\left[\left\| \mathbf{y}(\mathbf{x}_0) - \hat{\mathbf{y}}(\mathbf{x}_0 | \mathbf{c}(\mathbf{x}), \mathbf{V}^{*(k)}, \hat{\Theta}^{(k)}) \right\|^2 \middle| \mathbf{c}(\mathbf{x}), \mathbf{V}^{*(k)}, \hat{\Theta}^{(k)}, (\hat{\sigma}^{(k)})^2 \right] \middle| \mathbf{V}^{*(k)}, \hat{\Theta}^{(k)}, (\hat{\sigma}^{(k)})^2 \right\}, \tag{13}$$

with

$$\begin{aligned} \hat{y}\left(\mathbf{x}_0 | \mathbf{c}(\mathbf{x}), \mathbf{V}^{*(k)}, \hat{\Theta}^{(k)}\right) &= \mathrm{E}\left[y(x_0) \middle| \mathbf{c}(\mathbf{x}), \mathbf{V}^{*(k)}, \hat{\Theta}^{(k)}, (\hat{\sigma}^{(k)})^2 \right] \\ &= \mathbf{B}^{(k)} \hat{c}\left(\mathbf{x}_0 | \mathbf{c}(\mathbf{x}), \mathbf{V}^{*(k)}, \hat{\Theta}^{(k)}\right), \end{aligned}$$

Proposition

Suppose the expectations in (13) and (14) are taken with respect to the approximate predictive distribution (8). Then, for any $\boldsymbol{x} \in \boldsymbol{X} \backslash \boldsymbol{X}^{(k)}$

$$J(\boldsymbol{x}_0, \boldsymbol{x}) = (\hat{\sigma}^{(k)})^2 L + \sum_{i=1}^{p_k} (d_i^{(k)})^2 \hat{\sigma}_i^2 \left(x_0 | \boldsymbol{x}, \boldsymbol{v}_i^{(k)}, \hat{\theta}_i^{(k)} \right), \tag{14}$$

where $d_i^{(k)}$ *is the* i*-th largest singular value of* $\boldsymbol{Y}^{(k)}$*,*

$$\hat{\sigma}_i^2 \left(\boldsymbol{x}_0 | \boldsymbol{x}, \boldsymbol{v}_i^{(k)}, \hat{\theta}_i^{(k)} \right) = \frac{\rho_i^{(k)}(\boldsymbol{x}_0, \boldsymbol{x})}{\alpha_i + k} \left(\beta_i + \frac{\alpha_i + k}{\alpha_i + k - 1} \psi_i^{(k)} \right),$$

$$\rho_i^{(k)}(\mathbf{x}_0, \mathbf{x}) = 1 - \tilde{k}_i(\mathbf{x}_0, \mathbf{x})^T \tilde{K}_i^{-1}(\mathbf{x}) \tilde{k}_i(\mathbf{x}_0, \mathbf{x}),$$
$$\psi_i^{(k)} = (\mathbf{v}_i^{(k)})^T (\mathbf{K}_i^{(k)})^{-1} \mathbf{v}_i^{(k)},$$
$$\tilde{k}_i(\mathbf{x}_0, \mathbf{x}) = [K(\mathbf{x}_0, \mathbf{x}_1^{(k)}; \hat{\theta}_i^{(k)}), \ldots, K(\mathbf{x}_0, \mathbf{x}_k^{(k)}; \hat{\theta}_i^{(k)}), K(\mathbf{x}_0, \mathbf{x}; \hat{\theta}_i^{(k)})]^T,$$
$$\tilde{K}_i(\mathbf{x}) = \begin{bmatrix} \mathbf{K}_i^{(k)} & \mathbf{k}_i^{(k)}(\mathbf{x}) \\ \mathbf{k}_i^{(k)}(\mathbf{x})^T & 1 \end{bmatrix},$$

with $\mathbf{v}_i^{(k)}$ being the i-th column of $\mathbf{V}^{*(k)}$, for $i = 1, \ldots, p_k$, $\mathbf{x}_j^{(k)}$ being the j-th point of $\mathbf{X}^{(k)}$ for $j = 1, \ldots, k$, $\mathbf{K}_i^{(k)}$ being a $k \times k$ matrix with $K(\mathbf{x}_j^{(k)}, \mathbf{x}_l^{(k)}; \hat{\theta}_i^{(k)})$, as the (j, l)-th entry, and $\mathbf{k}_i^{(k)}(\mathbf{x}) = [K(\mathbf{x}, \mathbf{x}_1^{(k)}; \hat{\theta}_i^{(k)}), \ldots, K(\mathbf{x}, \mathbf{x}_k^{(k)}; \hat{\theta}_i^{(k)})]^T$.

J-Criterion

- Finding x_{k+1}^* by minimizing the J-criterion in Proposition 1 is equivalent to obtaining

$$x_{k+1}^* = \underset{x \in X \setminus \mathbf{x}^{(k)}}{\operatorname{argmin}} \left[\sum_{i=1}^{p_k} (d_i^{(k)})^2 \hat{\sigma}_i^2 \left(x_0 | \mathbf{x}, \mathbf{v}_i^{(k)}, \hat{\theta}_i^{(k)} \right) \right]. \tag{15}$$

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- This J-criterion is a generalization of the active learning Cohn (ALC) criterion (Cohn et al., 1996)

$$\int_{\mathbf{x}} \left[\int_y \left(\hat{y}(\mathbf{x}) - y(\mathbf{x}) \right)^2 dP(y|\mathbf{x}) \right] dP(\mathbf{x}),$$

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- We use L_2 norm discrepancy instead of the squared error.

Simulations

- We compare the performance of the three methods *full svdGP*, *naive local svdGP*, and *greedy local svdGP*

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- We compare the performance of the three methods *full svdGP*, *naive local svdGP*, and *greedy local svdGP*
- The performance is evaluated by comparing

$$\text{NMSPE}(x) = \frac{\sum_{t=1}^{L} \left(y_t(x) - \hat{y}_t(x)\right)^2}{\sum_{t=1}^{L} \left(y_t(x) - \bar{y}(x)\right)^2}, \tag{16}$$

where $y_t(x)$ is the real (scalar) response for input x at time t, $t = 1, \ldots, L$, $\hat{y}_t(x)$ is the corresponding model prediction, and $\bar{y}(x) = \sum_{t=1}^{L} y_t(\mathbf{x})/L$.

Example 1 (Forrester et al., 2008)

$$f(x,t) = (x_1 t - 2)^2 \sin(x_2 t - x_3), \tag{17}$$

where $x = (x_1, x_2, x_3)^T \in [4,10] \times [4,20] \times [1,7]$, and $t \in [1,2]$ is on a 200-point equidistant time-grid.

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- The training data is generated using a 10000-point random Latin hypercube design (LHD)
- The test data is generated using a 2000-point random LHD
- For local svdGP, use $n = 20, 40$, and $n_0 = \lceil n/4 \rceil, \lceil n/2 \rceil$

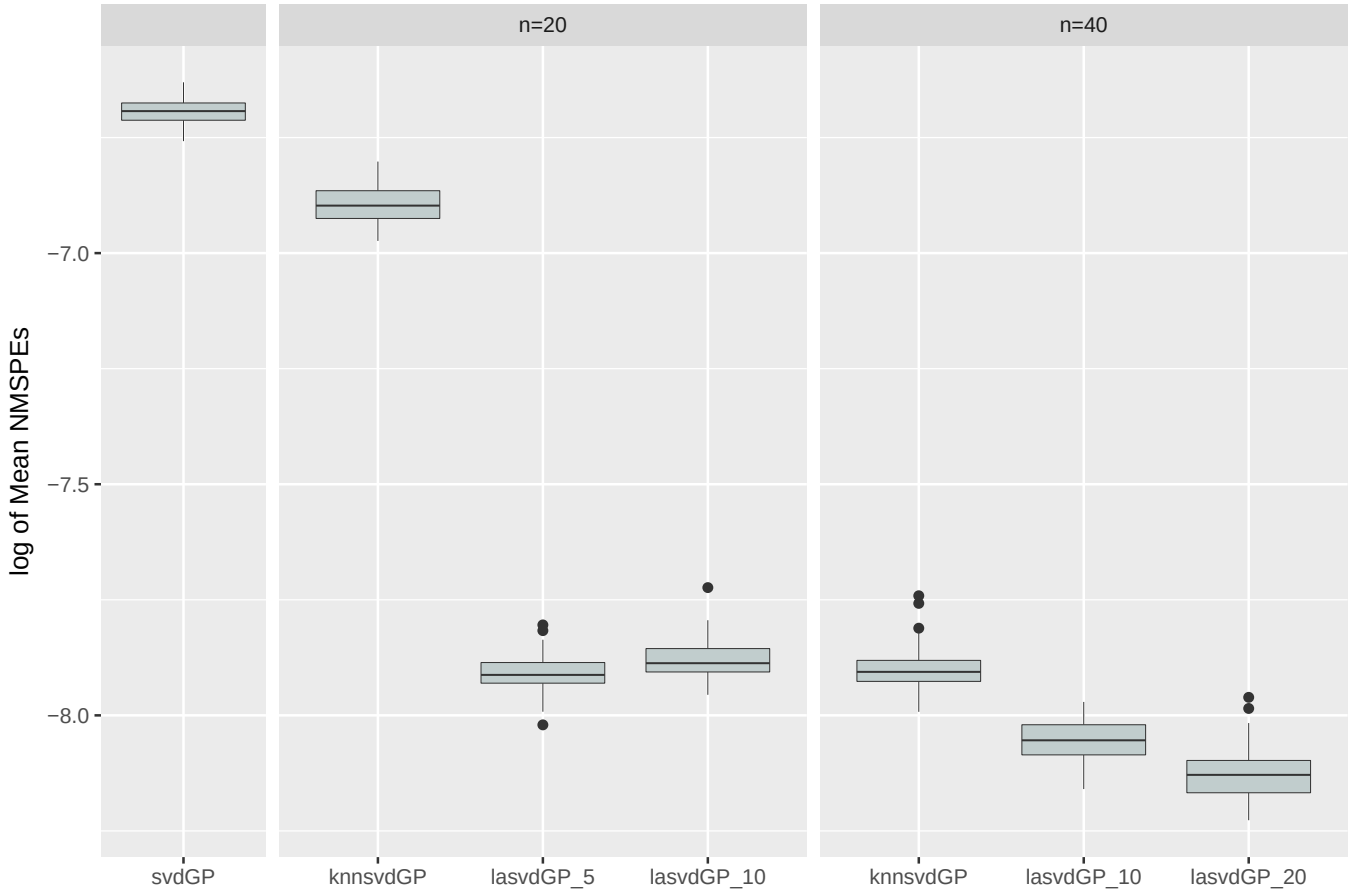

Figure: Boxplots of the log of mean NMSPE computed from 2,000 test points over 50 simulations for the computer simulator (17).

Example 2 (Bliznyuk et al.,2008)

$$f(x,t) = \frac{M}{\sqrt{Dt}} \exp\left(\frac{-s^2}{4Dt}\right) + \frac{M}{\sqrt{D(t-\tau)}} \exp\left(-\frac{(s-L)^2}{4D(t-\tau)}\right) I(\tau < t), \tag{18}$$

where $x = (M, D, L, \tau, s)^T$, M denotes the mass of pollutant spilled at each location, D is diffusion rate in the channel, L is location of the second spill, and τ is time of the second spill, $t \in [0.3, 60]$ is on a regular 200-point equidistant time grid,
$x \in [7, 13] \times [0.02, 0.12] \times [0.01, 3] \times [30.01, 30.295] \times [0, 3]$, .

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- For local svdGP, use $n = 30, 50$, and $n_0 = \lceil n/4 \rceil, \lceil n/2 \rceil$

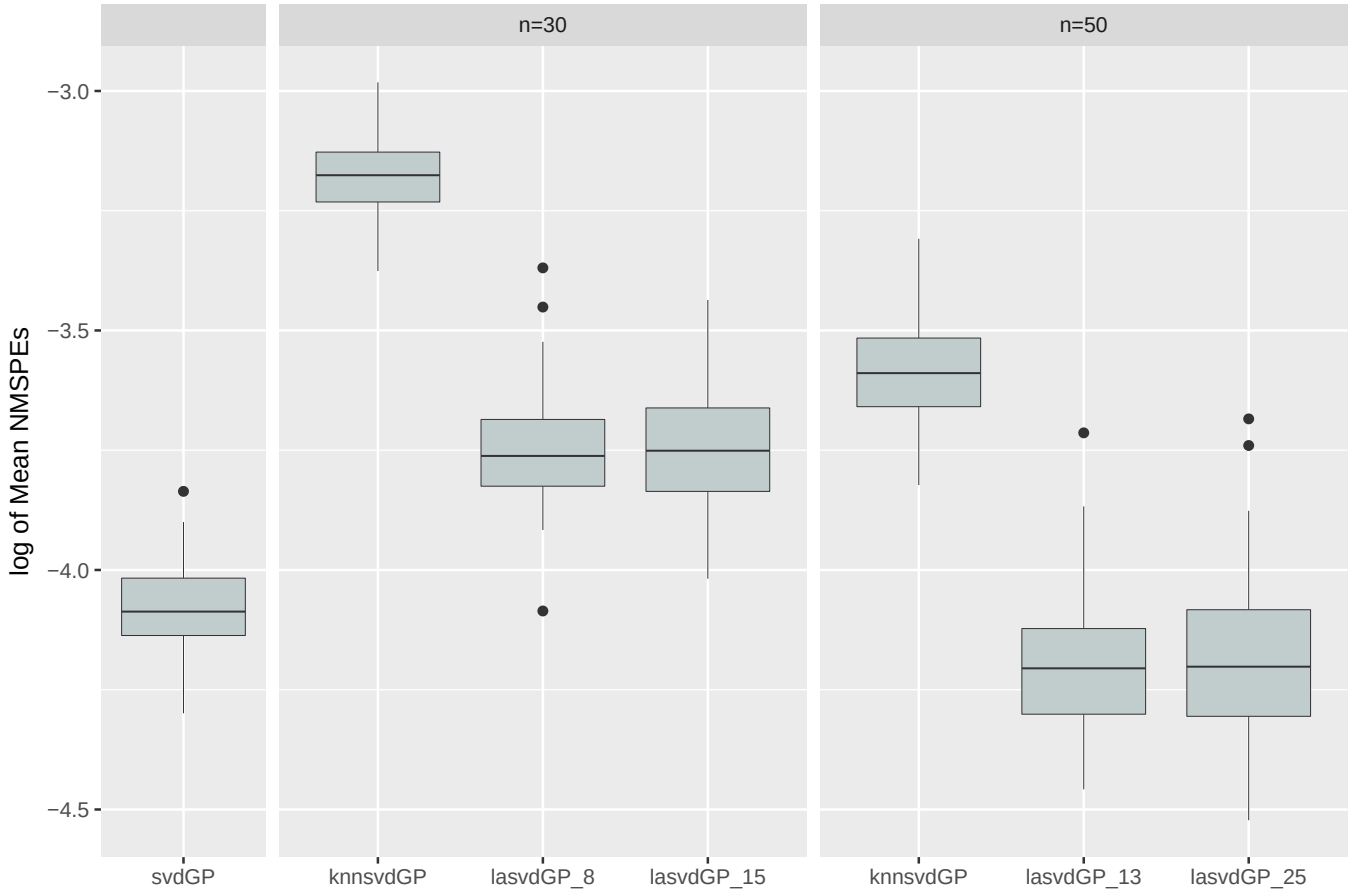

Figure: Boxplots of the log of mean NMSPE computed from 2,000 test points over 50 simulations for the computer simulator (19).

Example 3 (TDB simulator)

- The two-delay blowfly (TDB) model (Teismann et al., 2009) simulates European red mites (ERM) population dynamics under predator-prey interactions in apple orchards via numerically solving the Nicholson's blowfly differential equation (Gurney et al., 1980).

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- $N = 20,000$ training and $M = 10,000$ test points are generated using random LHDs for the emulation of this process.

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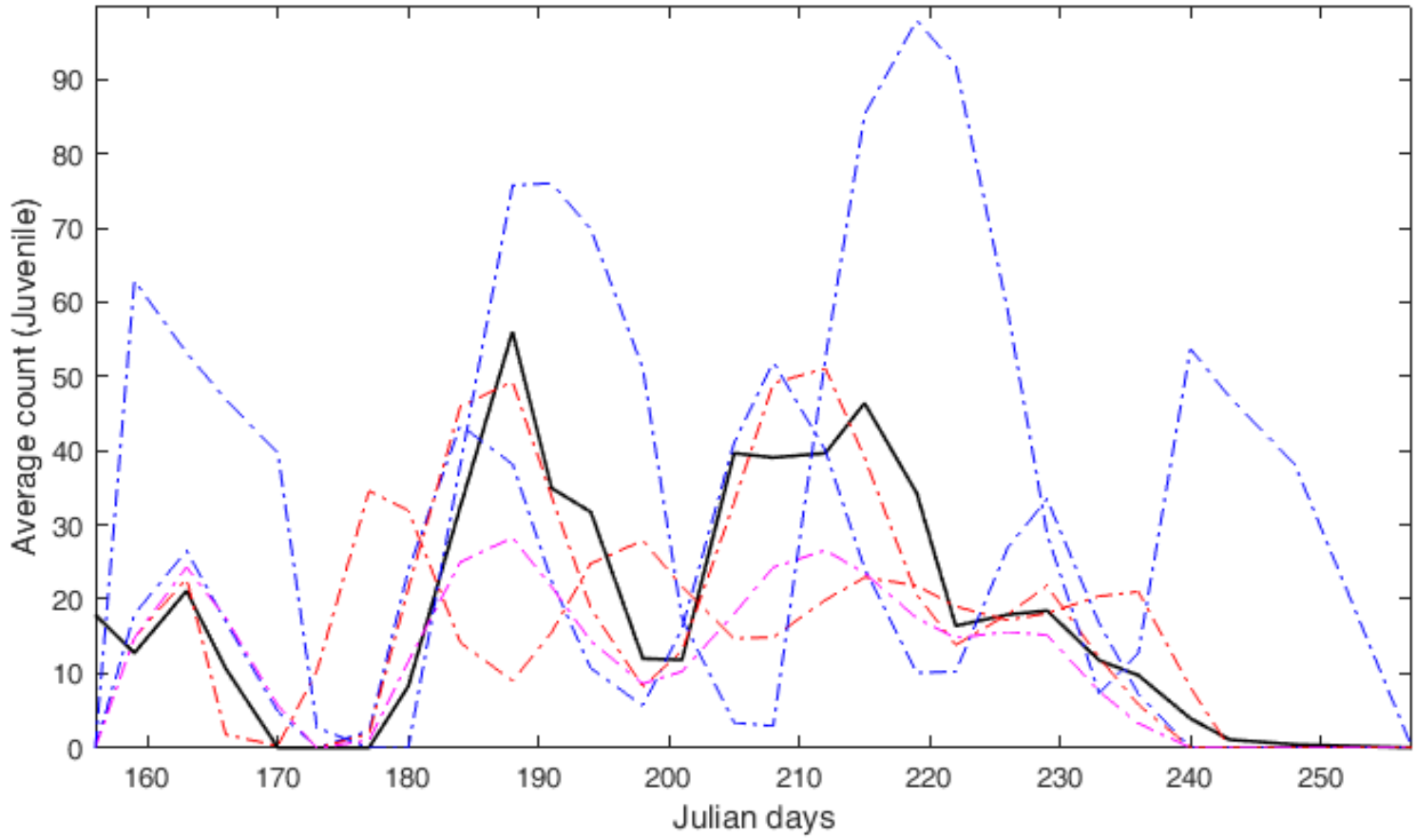

Figure: Juvenile ERM population dynamics as outputs of the TDB model at five different inputs. Black solid curve shows the field data and the other curves show the TDB outputs.

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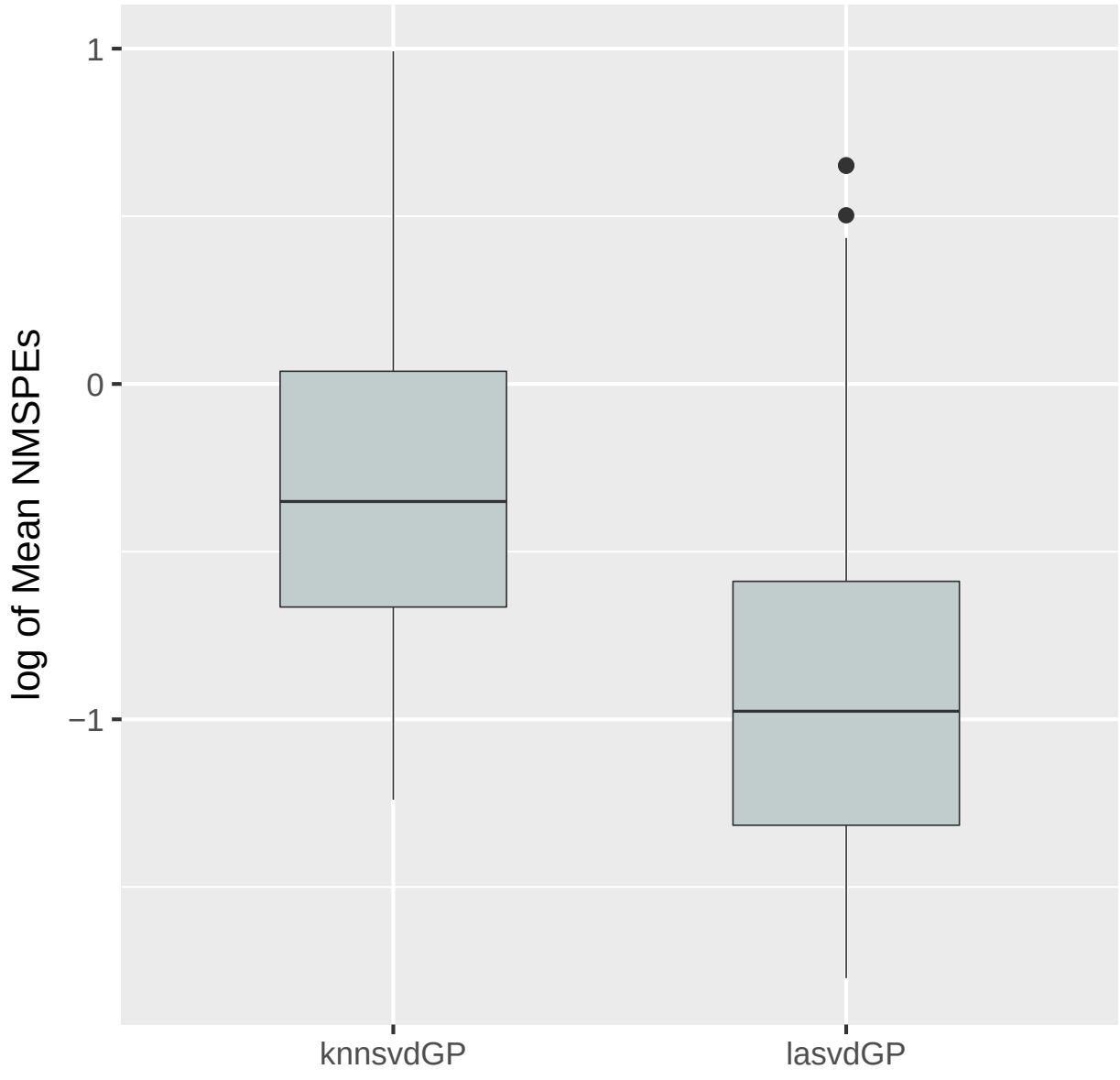

Inverse Problem

- Take a q-dimensional input $\mathbf{x}$ and return a time series response $\mathbf{y}(\mathbf{x}) = \{y(\mathbf{x}, t), t = 1, \ldots, L\}$ and consider a target field observation $\mathbf{z}_0$

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- Consider the following model, for $t = 0.5, 0.52, \ldots, 2.5$ and $x \in [0, 1]$,

$$y(x, t) = \frac{sin(10\pi t)}{2t} + |t - 1|^{(2+4x)}, \tag{19}$$

and let the true field data correspond to $x_0 = 0.5$.

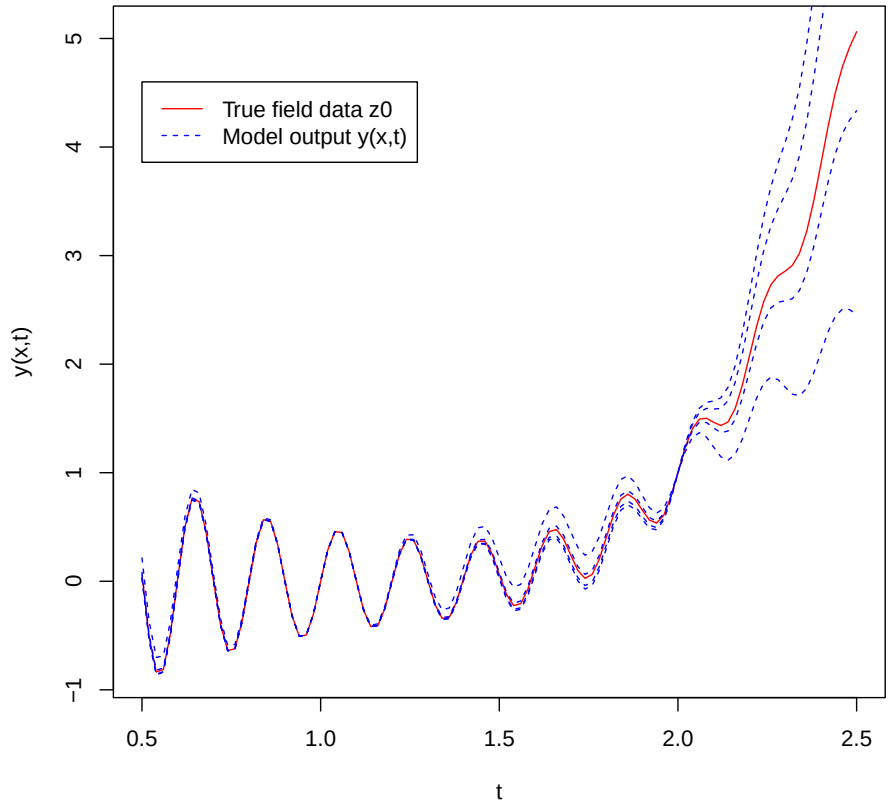

Figure: A few computer model outputs and the true field data for the computer simulator (19).

Existing Methods

- Scalarization by likelihood ratio statistic (Pratola et al., 2013)

$$\Delta(\mathbf{t}_i) = -2\log\left(\frac{L_r^*(\delta(\mathbf{t}_i))}{L_u^*(\delta(\mathbf{t}_i))}\right), \tag{20}$$

where $\delta(\mathbf{t}_i) = \mathbf{Y}^f - \mathbf{Y}_i^c$, $\mathbf{Y}^f = \eta(\theta) + \varepsilon$, and $\mathbf{Y}_i^c = \eta(\mathbf{t}_i)$, $\mathbf{t}_i$ is the ith input setting of calibration parameters.

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- Scalarization by L_2 discrepancy (Ranjan et al., 2016).

$$w(\mathbf{x}) = ||\mathbf{y}(\mathbf{x}) - \mathbf{z}_0|| = \sqrt{\frac{1}{L}\sum_{t=1}^{L} |\mathbf{y}(\mathbf{x}, t) - \mathbf{z}_0(t)|^2} \tag{21}$$

  and find the minimum of $w(\mathbf{x})$.

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$$\begin{aligned}
\hat{\mathbf{x}}^* &= \arg\min_{\mathbf{x}} E[||\mathbf{z}_0 - \mathbf{y}(\mathbf{x})||_2^2 | \mathbf{z}_0, \mathbf{Y}] \\
&= \arg\min_{\mathbf{x}} \sum_{i=1}^{p} d_i^2 [(\hat{c}_i(\mathbf{x}|\mathbf{v}_i, \hat{\theta}_i) - \hat{c}_{z_o,i})^2] + \sum_{i=1}^{p} d_i^2 [\hat{\sigma}_i^2(\mathbf{x}|\mathbf{v}_i, \hat{\theta}_i)].
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- The first term represents the L_2 distance between the posterior mean of the simulator response at $\mathbf{x}$ and the projection of $\mathbf{z}_0$ on $\mathcal{M}(\mathbf{B})$ with $\mathbf{B} = \mathbf{U}^*\mathbf{D}^*$; the second term represents the model uncertainty at $\mathbf{x}$.

Sequential Designs

Choose the follow-up design points by sequentially maximizing the criterion

$$\begin{aligned} \mathbf{x}_{new} &= \underset{\mathbf{x}}{\operatorname{argmax}}\, J_l(\mathbf{x}) \\ &= \underset{\mathbf{x}}{\operatorname{argmax}} \sum_{i=1}^{p} d_i^2 [\xi \hat{\sigma}_i^2(\mathbf{x}|\mathbf{v}_i, \hat{\theta}_i) - (\hat{c}_i(\mathbf{x}|\mathbf{v}_i, \hat{\theta}_i) - \hat{c}_{\mathbf{z}_0,i})^2] \end{aligned}$$

where ξ is a tuning parameter which controls the emphasis on reducing the model uncertainty.

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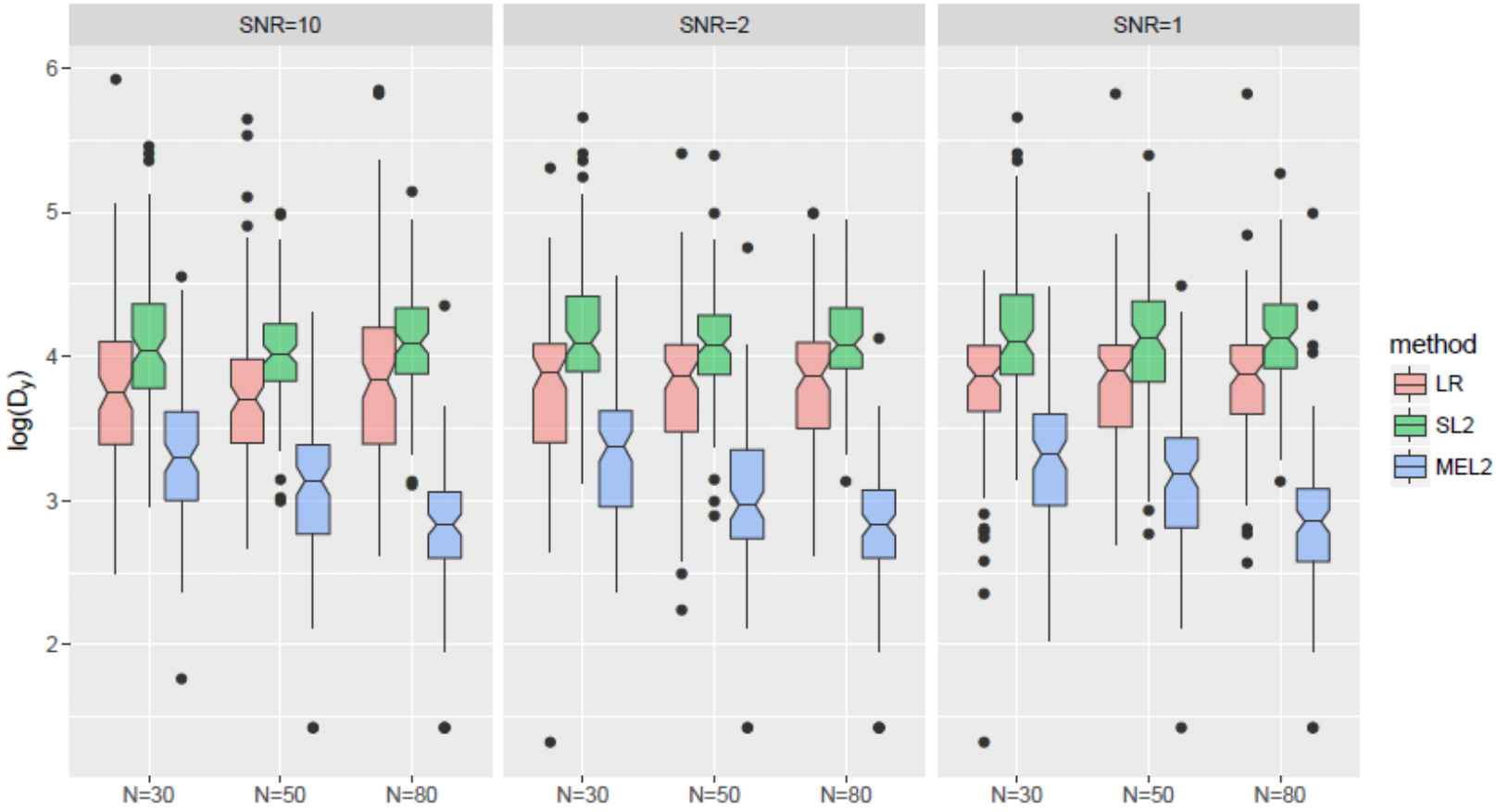
- For a pre-specified $\mathbf{x}^*$, compare using $D_{\mathbf{y}} = ||\mathbf{y}(\hat{\mathbf{x}}^*) - \mathbf{y}(\mathbf{x}^*)||_2$.

Consider again the environmental model studied by Bliznyuk et al. (2008).



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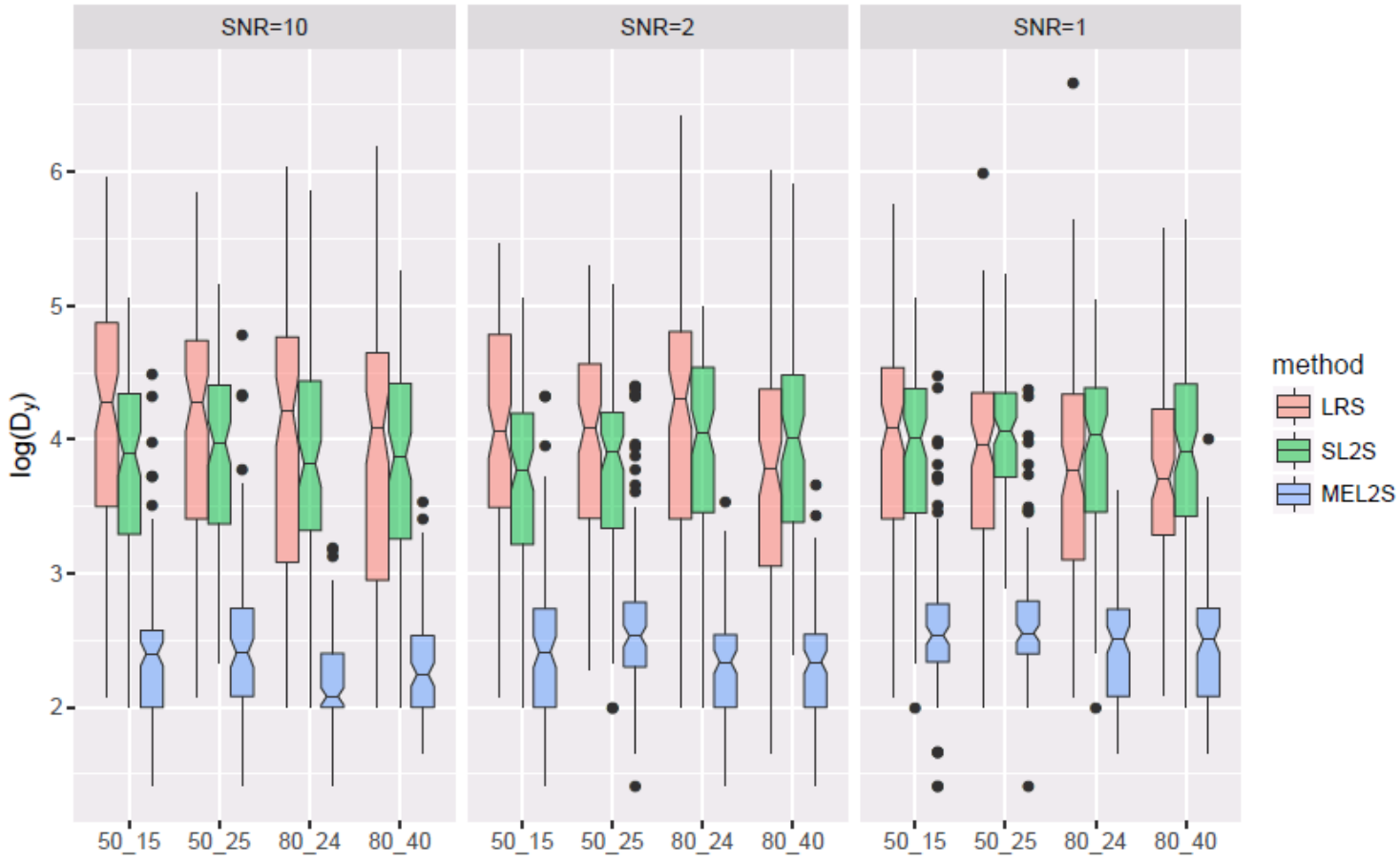

Concluding Remarks

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- Proposed a new design criterion to determine local neighbour sequentially
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Thank you! Q&A.