Some refinements of Dade's Projective Conjecture

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New Perspectives in Representation Theory of Finite Groups

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Schur indices

- I. Schur
- R. Brauer
- W. Feit

$\chi \in \mathrm{Irr}(G)$

F a field of characteristic zero

$\overline{\chi}$ the sum of all Galois conjugates of χ over F

$m_F(\chi)$ is the smallest number such that some module over F affords $\mathrm{m}_F(\chi)\overline{\chi}$

$m_F(\chi)$ can be any positive integer (Brauer)

Feit's Question: If G is quasi-simple, is $m_F(\chi) \leq 2$?

If G is perfect, $m_F(\chi)$ can be any positive integer (Turull)

Local Schur indices

$m_p(\chi) = \mathrm{m}_{\mathbb{Q}_p}(\chi)$ local Schur index of χ

Local Schur indices are unbounded even on characters in block of cyclic defect

Brauer groups

If $F(\chi) = F$, then $\mathrm{End}_{FG}(M)$ determines a unique element $[\chi]_F$ of $\mathrm{Br}(F)$

These elements are of unbounded order, even among local fields F and χ in blocks of cyclic defect

Dade's Projective Conjecture

Notation

p a prime.

$\mathbf{Q}_p$ field of p-adic numbers. $\overline{\mathbf{Q}_p}$ the algebraic closure of $\mathbf{Q}_p$.

G, H finite groups.

Z a p-subgroup of $\mathrm{Z}(G)$.

$\mathcal{N}(G, Z)$ the set of all sets C of p-subgroups of G such that each element of C strictly contains Z, C is totally ordered by inclusion, and all elements of C are normal subgroups of the largest element of C.

$\mathcal{N}(G, Z)/G$ a set of representatives of the orbits of G.

Let $\chi \in \mathrm{Irr}(H)$.

$\mathrm{codeg}(\chi) = |H|/\chi(1)$ the codegree of χ.

$\mathrm{d}(\chi)$ and $\mathrm{r}(\chi)$ the unique integers so that

$$\mathrm{codeg}(\chi) = p^{\mathrm{d}(\chi)}\,\mathrm{r}(\chi)$$

and p does not divide $\mathrm{r}(\chi)$. We call $\mathrm{d}(\chi)$ the p-defect of χ, and we call $\mathrm{r}(\chi)$ the p-residue of χ.

Let $\lambda \in \mathrm{Irr}(Z)$, B be a p-block of G, and let d be a non-negative integer. For H a subgroup of G with $Z \subseteq H$,

$$k(H, B, \lambda, d)$$

is the number of elements $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{d}(\psi) = d$, ψ is in a block that induces to B, and the restriction of ψ to Z contains λ as an irreducible constituent.

Dade Projective Conjecture

Let $\lambda \in \mathrm{Irr}(Z)$, B be a p-block of G, and let d be a non-negative integer. Assume that Z is not a defect group of B. Then

$$\sum_{C \in \mathcal{N}(G,Z)/G} (-1)^{|C|} k(\mathrm{N}_G(C), B, \lambda, d) = 0.$$

Britta Späth recently published a reduction theorem for Dade's Projective Conjecture.

Refinements

Refining k

F a field with $\mathbf{Q}_p \subseteq F \subseteq \overline{\mathbf{Q}_p}$.

$r \in \{1, \ldots, p-1\}$.

$$k(H, B, \lambda, d, r, F)$$

the number of elements $\psi \in \mathrm{Irr}(H)$ such that ψ is in a p-block that induces to B, ψ has p-defect d, ψ has p-residue congruent to $\pm r$ modulo p, the restriction of ψ to Z contains λ as an irreducible constituent, and $\mathbf{Q}_p(\psi) = F$.

Refined Conjecture (Dade, Isaacs, Navarro, Uno). Under the assumptions of Dade's Projective Conjecture we should also have

$$\sum_{C \in \mathcal{N}(G,Z)/G} (-1)^{|C|} k(\mathrm{N}_G(C), B, \lambda, d, r, F) = 0.$$

One then recovers the original Dade conjecture by addition over the new variables.

Conjecture (Boltje)

Let n be any integer. Assume that Z is not a defect group of B.

Then

$$\sum_{C \in \mathcal{N}(G,Z)_{\leq n}/G} (-1)^{n-|C|} k(\mathrm{N}_G(C), B, \lambda, d, r, F) \geq 0.$$

Further refining k

Pick some $s \in \mathrm{Br}(F)$. Then

$$k(H, B, \lambda, d, r, F, s)$$

is the number of elements $\psi \in \mathrm{Irr}(H)$ as before which, in addition, satisfy

$$[\psi]_F = s.$$

In particular, all the relevant ψ have the same Schur index $m_F(\psi)$ over F.

Refined Conjecture

Let n be any integer. Assume that Z is not a defect group of B.

Then

$$\sum_{C \in \mathcal{N}(G,Z)_{\leq n}/G} (-1)^{n-|C|} k(\mathrm{N}_G(C), B, \lambda, d, r, F, s) \geq 0.$$

Theorem

The Refined Conjecture holds whenever G is p-solvable.

Corollary

All the other refinements of the Dade Projective Conjecture hold for all p-solvable groups.

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Earlier work

G. R. Robinson (2,000)

A. Glesser (2,007)

Robinson's approach is indirect

It uses Külshammer-Puig on extensions of nilpotent blocks

Glesser uses a similar approach

The new proof

- Uses a direct approach via reduction theorems
- Does not use Külshammer-Puig on extensions of nilpotent blocks
- Does not assume special cases of the conjecture

Tools

A. Turull, Above the Glauberman correspondence (2008)

A. Turull, Inverse Glauberman-Isaacs correspondence and subnormal subgroups (2015)

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Character triple isomorphisms preserving rationality

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Reduction theorems

Theorem

Let D be a defect group for B. Suppose that N is a normal p'-subgroup of G. Then there exist $\theta_0 \in \mathrm{Irr}_D(N)$, $T = \tilde{I}_G(\theta_0, F)$, and $B_0 \in \mathrm{Bl}_p(T|\theta_0)$ satisfying all the following.

1. *$B = B_0^G$, $B \in \mathrm{Bl}_p(G|\theta_0)$, and D is a defect group of B_0.*
2. *Let $C_1 \in \mathcal{N}(G, Z)$, and let C_1^G be the set of G conjugates of C_1. Then*

$$k(\mathrm{N}_G(C_1), B, \lambda, d, r, F, s) = \sum_{C \in \left(C_1^G \cap \mathcal{N}(T,Z)\right)/T} k(\mathrm{N}_T(C), B_0, \lambda, d, r, F, s).$$

Theorem

Let N be a normal p'-subgroup of G, and let M be a normal subgroup of G such that $ZN \subseteq M$ and M/N is a p-group. Let P be a Sylow p-subgroup of M. Let $\theta \in \mathrm{Irr}(N)$ be P-invariant, and let $\eta \in \mathrm{Irr}(\mathrm{C}_N(P))$ be the Glauberman correspondent of θ with respect to the action of P so that $\eta = \pi(P, N)(\theta)$. Assume that $G = \tilde{I}_G(\theta, F)$, and that there is a single p-block block B of G above θ, so that $\{B\} = \mathrm{Bl}_p(G|\theta)$. Let $B_0 \in \mathrm{Bl}_p(\mathrm{N}_G(P)|\eta)$. Then, $\{B_0\} = \mathrm{Bl}_p(\mathrm{N}_G(P)|\eta)$, every defect group of B_0 is a defect group of B, $B_0^G = B$, and for every $C \in \mathcal{N}(\mathrm{N}_G(P), Z)$, we have

$$(1)\quad k(\mathrm{N}_G(C), B, \lambda, d, r, F, s) = k(\mathrm{N}_G(C) \cap \mathrm{N}_G(P), B_0, \lambda, d, r, F, s).$$

Theorem

The Refined Conjecture holds whenever G is p-solvable.