Perfect Isometries and Basic Sets

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Basic Sets: Motivation, Definition

Let G be a finite group and $\mathrm{Irr}(G)$ be the set of irreducible complex characters of G.
Let p be a prime (dividing $|G|$), and let $\mathcal{C}$ be the set of p-regular elements of G.

One of the main problems in modular representation theory is to find the (p-modular) decomposition matrix D of G.
For example, the decomposition matrices are not known for the symmetric group $\mathfrak{S}_n$ or the alternating group $\mathfrak{A}_n$.

Basic sets can sometimes help solving this problem, or at least reducing it.

For each $\chi \in \mathbb{C}\,\mathrm{Irr}(G)$, we define a class function $\chi^{\mathcal{C}}$ of G by letting

$$\chi^{\mathcal{C}}(g) = \begin{cases} \chi(g) & \text{if } g \in \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases}$$

We call *p-basic set for G* any subset $\mathcal{B} \subset \mathrm{Irr}(G)$ such that the family $\mathcal{B}^{\mathcal{C}} = \{\chi^{\mathcal{C}}, \chi \in \mathcal{B}\}$ is a $\mathbb{Z}$-basis for the $\mathbb{Z}$-module generated by $\mathrm{Irr}^{\mathcal{C}}(G) = \{\chi^{\mathcal{C}}, \chi \in \mathrm{Irr}(G)\}$.

In particular, $|\mathcal{B}|$ is the number of p-regular conjugacy classes of G.

If $\mathcal{B}$ is a p-basic set for G, and if we write $\chi^{\mathcal{C}} = \sum_{\psi \in \mathcal{B}} n_{\chi\psi}\psi^{\mathcal{C}}$ ($\chi \in \mathrm{Irr}(G)$, $n_{\chi\psi} \in \mathbb{Z}$) and $N_{\mathcal{B}} = ((n_{\chi\psi}))_{\chi \in \mathrm{Irr}(G), \psi \in \mathcal{B}}$, and $D_{\mathcal{B}}$ for the (square) sub-matrix of D whose rows correspond to $\mathcal{B}$, then we have

$$D = N_{\mathcal{B}} D_{\mathcal{B}}.$$

One can define the notion of p-basic set for a p-block of G, and one shows easily that, if each p-block b of G has a p-basic set $\mathcal{B}_b$, then the union of the $\mathcal{B}_b$'s is a p-basic set for G.

State of the Art

Hiss Conjecture (HC, 19??): If G is a finite group and p is any prime, then G has a p-basic set.

HC is true if $G = G_2(q)$ (Hiss) or $G = SU_3(q^2)$ (Geck) and $p \nmid q$ (and this led to constructing the decomposition matrices).

HC is true if $G = \mathfrak{S}_n$ (James-Kerber), if $G = GL_2(q)$, $GL_3(q)$ or $GL_4(q)$ and $p|q$ (Brunat), if G is of Lie type, p is non-defining, and G satisfies certain additional properties (Geck, Geck-Hiss), or if G is p-solvable (Fong-Swan).

HC is true if $G = \tilde{\mathfrak{S}}_n$, a double Schur cover of $\mathfrak{S}_n$, and $p = 2$ (Bessenrodt-Olsson).

HC is also true if $G = \mathfrak{A}_n$ (any p), and if $G = \tilde{\mathfrak{S}}_n$ or $G = \tilde{\mathfrak{A}}_n$ and p is odd (Brunat-Gramain).

Semi-Direct Products

The first step to show the existence of p-basic sets for p-solvable groups is the case of semi-direct products.

If $G = P \rtimes Q$, where P is a p-group and Q is a p'-group, then the set $\mathcal{B}$ of irreducible characters of G with P in their kernel is a p-basic set for G.

We have $\mathcal{B} = \{\psi \circ \pi \mid \psi \in \mathrm{Irr}(Q))\}$, where $\pi\colon G \longrightarrow Q$ is the canonical surjection.
Also, all elements of Q are p-regular, and the conjugacy classes of p-regular elements of G are in bijection with the conjugacy classes of Q.

In this case, the matrix $N_{\mathcal{B}} = ((n_{\chi\psi}))_{\chi\in\mathrm{Irr}(G),\psi\in\mathrm{Irr}(Q)}$ is known in theory. We simply have

$$n_{\chi\psi} = \langle \mathrm{Res}_Q^G(\chi), \psi\rangle_Q$$

Perfect Isometries

If G and G' are finite groups, $\mathcal{C}$ and $\mathcal{C}'$ are the sets of p-regular elements of G and G' respectively, and if b and b' are unions of p-blocks of G and G' respectively, then a perfect isometry between b and b' is an isometry $\mathcal{I}\colon \mathbb{C}\,\mathrm{Irr}(b) \longrightarrow \mathbb{C}\,\mathrm{Irr}(b')$ such that

(1) $\mathcal{I}(\mathbb{Z}\,\mathrm{Irr}(b)) = \mathbb{Z}\,\mathrm{Irr}(b')$ and
(2) for all $\chi \in \mathrm{Irr}(b)$, we have $\mathcal{I}(\chi^{\mathcal{C}}) = (\mathcal{I}(\chi))^{\mathcal{C}'}$.

Note that Broué's perfect isometries are also perfect isometries in this sense.

Whenever $\chi \in \mathrm{Irr}(b)$, we let $\mathrm{I}(\chi) = \mathcal{I}(\chi)$ if $\mathcal{I}(\chi) \in \mathrm{Irr}(b')$, and $\mathrm{I}(\chi) = -\mathcal{I}(\chi)$ otherwise. In this way, I gives a bijection between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(b')$.

Proposition: If $\mathcal{I}$ is a perfect isometry between b and b', and $\mathcal{B}$ is a p-basic set for b, then $\mathrm{I}(\mathcal{B}) = \{\mathrm{I}(\chi), \chi \in \mathcal{B}\}$ is a p-basic set for b'.

Symmetric Group

The irreducible complex characters of the symmetric group $\mathfrak{S}_n$ are canonically labeled by the partitions of n.

We have $\mathrm{Irr}(\mathfrak{S}_n) = \{\chi_\lambda \mid \lambda \vdash n\}$, and two characters χ_λ and χ_μ of $\mathfrak{S}_n$ belong to the same p-block if and only if λ and μ have the same p-core (Nakayama Conjecture).

In particular, if a block b of $\mathfrak{S}_n$ corresponds to the p-core $\gamma \vdash r$, then $n = pw + r$, where w is the p-weight of b.

Theorem [Enguehard, 1990]: If b and b' are p-blocks of the symmetric groups $\mathfrak{S}_m$ and $\mathfrak{S}_n$ of the same p-weight, then b and b' are perfectly isometric.

It is therefore enough to consider the principal p-block of $\mathfrak{S}_{pw}$.

Abelian Defect

The principal p-block b of $\mathfrak{S}_{pw}$ has Abelian defect if and only if $w < p$.

If so, b has defect group $\mathbb{Z}_p^w$, and there is a perfect isometry between b and its Brauer correspondent b_0 in $N = N_{\mathfrak{S}_{pw}}(\mathbb{Z}_p^w)$ (Rouquier, 1994).

Now $N \cong N_{\mathfrak{S}_p}(\mathbb{Z}_p) \wr \mathfrak{S}_w$, and, as $N_{\mathfrak{S}_p}(\mathbb{Z}_p) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$, we get

$$N \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}) \wr \mathfrak{S}_w \cong \mathbb{Z}_p^w \rtimes (\mathbb{Z}_{p-1} \wr \mathfrak{S}_w),$$

and $\mathbb{Z}_{p-1} \wr \mathfrak{S}_w$ is a p'-group since $w < p$.

In fact, we have $\mathrm{Irr}(b_0) = \mathrm{Irr}(N)$, and so b_0 and b have a p-basic set.

Non-Abelian Defect

In order to force the previous argument to work when $w \geq p$, we generalize things a bit:

If G is a finite group and $\mathcal{C}$ is a union of conjugacy classes of G, and if $b \subset \mathrm{Irr}(G)$, then we call $\mathcal{C}$-basic set for b any subset $\mathcal{B} \subset b$ such that the family $\mathcal{B}^{\mathcal{C}} = \{\chi^{\mathcal{C}}, \chi \in \mathcal{B}\}$ is a $\mathbb{Z}$-basis for the $\mathbb{Z}$-module generated by $b^{\mathcal{C}} = \{\chi^{\mathcal{C}}, \chi \in b\}$.

If now $\mathcal{C}'$ is a union of conjugacy classes of G', and if $b' \subset \mathrm{Irr}(G')$, then a $(\mathcal{C}, \mathcal{C}')$-perfect isometry between b and b' is an isometry $\mathcal{I} \colon \mathbb{C}b \longrightarrow \mathbb{C}b'$ such that

(1) $\mathcal{I}(\mathbb{Z}b) = \mathbb{Z}b'$ and
(2) for all $\chi \in b$, we have $\mathcal{I}(\chi^{\mathcal{C}}) = (\mathcal{I}(\chi))^{\mathcal{C}'}$.

In analogy with the p-regular case, the image of a $\mathcal{C}$-basic set for b via a $(\mathcal{C}, \mathcal{C}')$-perfect isometry gives a $\mathcal{C}'$-basic set for b'.

Non-Abelian Defect

Now let b be the principal p-block of $\mathfrak{S}_{pw}$, and consider again $N \cong \mathbb{Z}_p^w \rtimes (\mathbb{Z}_{p-1} \wr \mathfrak{S}_w)$ inside $\mathfrak{S}_{pw}$.

Let $\mathcal{C}$ be the set of p-regular elements of $\mathfrak{S}_{pw}$, and let $\mathcal{C}'$ consist of those elements of N whose (p)-cycles only come from $\mathbb{Z}_{p-1} \wr \mathfrak{S}_w$. Thus, if $w < p$, $\mathcal{C}'$ is just the set of p-regular elements of N.

Theorem [Brunat-G, 2010]: There is a $(\mathcal{C}, \mathcal{C}')$-perfect isometry between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(N)$, and the irreducible characters of N with $\mathbb{Z}_p^w$ in their kernel form a $\mathcal{C}'$-basic set for $\mathrm{Irr}(N)$. In particular, b has a p-basic set.

Putting blockwise results back together, this gives a p-basic for $\mathfrak{S}_n$, which, unlike that of James and Kerber, can be used to produce a p-basic set for $\mathfrak{A}_n$ (easily if p is odd).

Covering Groups of $\mathfrak{S}_n$ and $\mathfrak{A}_n$

There is a nonsplit exact sequence

$$1 \longrightarrow \langle z \rangle \longrightarrow \tilde{\mathfrak{S}}_n \xrightarrow{\pi} \mathfrak{S}_n \longrightarrow 1,$$

where $\langle z \rangle = Z(\tilde{\mathfrak{S}}_n) \cong \mathbb{Z}/2\mathbb{Z}$.

$\pi^{-1}(\mathfrak{A}_n) = \tilde{\mathfrak{A}}_n$ is **the** 2-fold covering group of $\mathfrak{A}_n$.

Irreducible complex characters of $\tilde{\mathfrak{S}}_n$ (resp. $\tilde{\mathfrak{A}}_n$) are either directly lifted from $\mathfrak{S}_n$ (resp. $\mathfrak{A}_n$), or are faithful, and then called spin characters.

If p is an odd prime, then each p-block of $\tilde{\mathfrak{S}}_n$ or $\tilde{\mathfrak{A}}_n$ contains

- either **no** spin character, whence is a p-block of $\mathfrak{S}_n$ or $\mathfrak{A}_n$,
- or **only** spin characters, and is called a spin block.

For any $G \leq \tilde{\mathfrak{S}}_n$, we write $\mathrm{SI}(G)$ for the set of irreducible spin characters of G.

Perfect Isometries Between Blocks of Covering Groups

Spin characters of $\tilde{\mathfrak{S}}_n$ and $\tilde{\mathfrak{A}}_n$ are labelled by the bar-partitions of n, i.e. the partitions of n in distinct parts.

If p is odd, then two spin characters of $\tilde{\mathfrak{S}}_n$ or $\tilde{\mathfrak{A}}_n$ belong to the same p-block if and only if the bar-partitions labelling them have the same $\bar{p}$-core (Morris Conjecture).

In particular, each spin p-block b of $\tilde{\mathfrak{S}}_n$ or $\tilde{\mathfrak{A}}_n$ has a $\bar{p}$-weight w and a sign $\sigma(b)$.

Theorem [Brunat-G, 2017]: Suppose p is odd, and b and b' are spin p-blocks of $\tilde{\mathfrak{S}}_n$ and $\tilde{\mathfrak{S}}_m$ with the same $\bar{p}$-weight. Then

- If $\sigma(b) = \sigma(b')$, then b and b' are perfectly isometric.
- If $\sigma(b) \neq \sigma(b')$, then b covers a unique spin block b^* of $\tilde{\mathfrak{A}}_n$, and b^* and b' are perfectly isometric.

Broué's Perfect Isometry Conjecture

For $w > 0$, let b be the principal spin p-blocks of $\tilde{\mathfrak{S}}_{pw}$ (p odd), and consider again $N = P \rtimes L$ inside $\mathfrak{S}_{pw}$, where $P \cong \mathbb{Z}_p^w$ and $L \cong \mathbb{Z}_{p-1} \wr \mathfrak{S}_w$.

Now $\tilde{N} = \pi^{-1}(N) \leq \tilde{\mathfrak{S}}_{pw}$ can be written as $\tilde{N} = Q \rtimes \tilde{L}$, where $\tilde{L} = \pi^{-1}(L)$ and $Q \cong P$. (In fact, $\pi^{-1}(P) = \langle z \rangle \times Q$.)

If $w < p$, then $\tilde{N}$ is the normalizer of Sylow p-subgroups of $\tilde{\mathfrak{S}}_{pw}$, and it has a single spin p-block.

Theorem [Livesey, 2016]: If $w < p$, then there is a perfect isometry between b and $\mathrm{SI}(\tilde{N})$,

Generalizing Livesey's Result

We have $\widetilde{N} = Q \rtimes \widetilde{L} \leq \widetilde{\mathfrak{S}}_{pw}$.

Let $\mathcal{C}$ be the sets of p-regular elements of $\widetilde{\mathfrak{S}}_{pw}$ and $\mathcal{C}'$ be the union of conjugacy classes of $\widetilde{N}$ which have a representative in $\widetilde{L}$.

Theorem [Brunat-G, 201?]: If p is odd and $w > 0$, then

- The set of irreducible spin characters of $\widetilde{N}$ with Q in their kernel is a $\mathcal{C}'$-basic set for $\mathrm{SI}(\widetilde{N})$. It is parametrized by $\frac{p-1}{2}$-quotients of w.
- There is a $(\mathcal{C}, \mathcal{C}')$-perfect isometry between b and $\mathrm{SI}(\widetilde{N})$.

In particular, b has a p-basic set.

A similar argument shows the the principal spin p-block of $\widetilde{\mathfrak{A}}_n$ has a p-basic set.

From this, we deduce that $\widetilde{\mathfrak{S}}_n$ and $\widetilde{\mathfrak{A}}_n$ have p-basic sets for any n.

Consequences

We can now recover easily some results of Olsson (1992).

Denote by ε the unique non-trivial linear character of $\tilde{\mathfrak{S}}_n$ with $\tilde{\mathfrak{A}}_n$ in its kernel (so that $\varepsilon = \mathrm{sgn} \circ \pi$, and $\tilde{\mathfrak{A}}_n = \ker(\varepsilon)$).
A spin character χ of $\tilde{\mathfrak{S}}_n$ is self-associate if $\varepsilon \otimes \chi = \chi$.

Now let b be a spin p-block of $\tilde{\mathfrak{S}}_n$ of weight w, and let $\mathcal{B}$ be the p-basic set for b obtained above.

The characters in $\mathcal{B}$ are labelled by the $(p-1)/2$-quotients of w, and each such quotient labels 2 spin characters in b, or each labels 1 spin character in b. This only depends on w.

In particular, all the characters in $\mathcal{B}$ are self-associate or none is, and this gives a formula for the number of irreducible Brauer characters in b (and for those in b^*).

Because $\mathcal{B}$ is ε-stable, we also obtain that all irreducible Brauer characters in b are self-associate, or none is.

Thank you!

(Banff, 18 March 2014)