# Zeta functions of alternate mirror Calabi-Yau pencils

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# Collaborators

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# Five Interesting Quartics in $\mathbb{P}^3$

| Family | Equation |
|---|---|
| $F_4$ (Fermat/Dwork) | $x_0^4 + x_1^4 + x_2^4 + x_3^4$ |
| $F_2L_2$ | $x_0^4 + x_1^4 + x_2^3x_3 + x_3^3x_2$ |
| $F_1L_3$ (Klein-Mukai) | $x_0^4 + x_1^3x_2 + x_2^3x_3 + x_3^3x_1$ |
| $L_2L_2$ | $x_0^3x_1 + x_1^3x_0 + x_2^3x_3 + x_3^3x_2$ |
| $L_4$ | $x_0^3x_1 + x_1^3x_2 + x_2^3x_3 + x_3^3x_0$ |

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## Warnings

- These quartics are not isomorphic.

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## Warnings

- These quartics are not isomorphic.
- These quartics are not Fourier-Mukai partners.
- These quartics are not derived equivalent.

# Counting Points

| Prime | $F_4$ | $F_2L_2$ | $F_1L_3$ | $L_2L_2$ | $L_4$ |
|---|---|---|---|---|---|
| 5 | 0 | 20 | 30 | 80 | 40 |

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| 7 | 64 | 50 | 64 | 64 | 78 |

# Counting Points

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| 5 | 0 | 20 | 30 | 80 | 40 |
| 7 | 64 | 50 | 64 | 64 | 78 |
| 11 | 144 | 122 | 144 | 144 | 254 |
| 13 | 128 | 180 | 206 | 336 | 232 |
| 17 | 600 | 328 | 294 | 600 | 328 |
| 19 | 400 | 362 | 400 | 400 | 438 |
| 23 | 576 | 530 | 576 | 576 | 622 |
| 29 | 768 | 884 | 1116 | 1232 | 1000 |
| 31 | 1024 | 962 | 1024 | 1024 | 1334 |
| 37 | 1152 | 1300 | 1374 | 1744 | 1448 |

Equality holds $\pmod{p}$ for all $p$ in this table.

# Counting Points on Pencils

- We can add the deforming monomial $-4\psi xyzw$ to each of our quartics to obtain pencils of quartics $X_{\diamond,\psi}$.

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- We can count the number of points on $X_{\diamond,\psi}$ over $\mathbb{F}_p$ for $0 \leq \psi < p$.
- For each $\psi$, the point counts on $X_{\diamond,\psi}$ agree $\pmod{p}$.

# The Zeta Function

We can organize point-count information in a generating function. Let $X/\mathbb{F}_q$ be an algebraic variety over the finite field of $q = p^s$ elements.

### Definition

The zeta function of $X$ is

$$Z(X/\mathbb{F}_q, T) := \exp\left(\sum_{s=1}^{\infty} \#X(\mathbb{F}_{q^s})\frac{T^s}{s}\right) \in \mathbb{Q}[[T]].$$

# Dwork and the Weil Conjectures

- $Z(X/\mathbb{F}_q, T)$ is rational
- We can factor $Z(X/\mathbb{F}_q, T)$ using polynomials with integer coefficients:

$$Z(X/\mathbb{F}_q, T) = \frac{\prod_{j=1}^{n} P_{2j-1}(T)}{\prod_{j=0}^{n} P_{2j}(T)},$$

- $\dim X = n$
- $P_0(t) = 1 - T$ and $P_{2n}(T) = 1 - p^n T$
- For $1 \leq j \leq 2n-1$, $\deg P_j(T) = b_j$, where $b_j = \dim H^j_{dR}(X)$.

# Projective Hypersurfaces

For a smooth projective hypersurface $X$ in $\mathbb{P}^n$, we have

$$Z(X, T) = \frac{P_X(T)^{(-1)^n}}{(1-T)(1-qT)\cdots(1-q^{n-1}T)},$$

with $P_X(T) \in \mathbb{Q}[T]$.

# Calabi-Yau Manifolds

We can define an $n$-dimensional Calabi-Yau manifold as a simply connected, smooth . . .

- Variety with trivial canonical bundle
- Ricci-flat Kähler-Einstein manifold
- Kähler manifold with a unique (up to scaling) nonvanishing holomorphic $n$-form

Calabi-Yau 2-folds are also known as K3 surfaces.

# A Pair of K3 Surface Examples

Let $p = 41$. Using Costa's code, we find:

$X : x_0^4 + x_1^4 + x_2^4 + x_3^4$

$P_X = (1 - 41T)^{18}(1 - 41T)(1 - 18T + 41^2 T^2)$

$X : x_0^4 + x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1$

$P_X = (1 - 41T)^3 (1 + 41T)^3 (1 - 41T)(1 - 18T + 41^2 T^2)(1 + 41^4 T^4)^3$

This factor is also preserved for our other families.

# The Unit Root

If $X$ is a Calabi-Yau hypersurface in $\mathbb{P}^n$, $P_X$ has at most one root that is a $p$-adic unit, termed the unit root. The value of this root determines $\#X(\mathbb{F}_q) \pmod{q}$.

# Why?

The arithmetic patterns we observe are a consequence of mirror symmetry.

# Mirror Symmetry

Physicists say . . .

- Calabi-Yau manifolds appear in pairs $(V, V^\circ)$.
- The universes described by $M_{3,1} \times V$ and $M_{3,1} \times V^\circ$ have the same observable physics.

Mathematicians say . . .

- Calabi-Yau manifolds appear in paired families $(V_\alpha, V_\alpha^\circ)$.
- Mirror symmetry interchanges deformations of complex and Kähler structures.

# Greene-Plesser Mirror Symmetry

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- Consider the Fermat quintic pencil $X_\psi$ given by

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0$$

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- The pencil admits a group action by $(\mathbb{Z}/5\mathbb{Z})^3$
- Taking the quotient by the group action and resolving singularities yields the mirror family $Y_\psi$

# Counting Deformation Parameters

- Smooth quintics in $\mathbb{P}^4$ have many complex deformation parameters
- The mirror family $Y_\psi$ is a one-parameter family

# The Hodge Diamond

## Calabi-Yau Threefolds

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1}(V) & & 0 & \\
1 & & h^{2,1}(V) & & h^{2,1}(V) & & 1 \\
 & 0 & & h^{1,1}(V) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

# Hodge diamond for the quintic and its mirror

**Smooth quintics**

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 1 | | | |
| | | 0 | | 0 | | |
| | 0 | | 1 | | 0 | |
| 1 | | 101 | | 101 | | 1 |
| | 0 | | 1 | | 0 | |
| | | 0 | | 0 | | |
| | | | 1 | | | |

**$Y_\psi$**

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 1 | | | |
| | | 0 | | 0 | | |
| | 0 | | 101 | | 0 | |
| 1 | | 1 | | 1 | | 1 |
| | 0 | | 101 | | 0 | |
| | | 0 | | 0 | | |
| | | | 1 | | | |

# Arithmetic Mirror Symmetry?

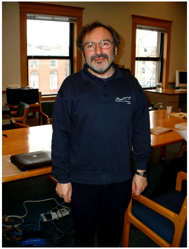

Figure: Philip Candelas

Figure: Xenia de la Ossa

Figure: Fernando Rodriguez Villegas

# Arithmetic Mirror Symmetry for Threefolds

- If $X$ and $Y$ are mirror Calabi-Yau threefolds, we can expect a relationship between $Z(X/\mathbb{F}_q, T)$ and $Z(Y/\mathbb{F}_q, T)$ due to the interchange of Hodge numbers.

# Arithmetic Mirror Symmetry for Threefolds

- If $X$ and $Y$ are mirror Calabi-Yau threefolds, we can expect a relationship between $Z(X/\mathbb{F}_q, T)$ and $Z(Y/\mathbb{F}_q, T)$ due to the interchange of Hodge numbers.
- Candelas, de la Ossa, and Rodriguez Villegas showed that for the Fermat quintic pencil $X_\psi$ and the Greene-Plesser mirror $Y_\psi$, $P_{X_\psi}$ and $P_{Y_\psi}$ share a common factor of degree 4.

# Greene-Plesser Mirror for Quartics in $\mathbb{P}^3$

- Start with all smooth quartics in $\mathbb{P}^3$
- Consider the Fermat pencil $X_\psi : x^4 + y^4 + z^4 + w^4 - 4\psi xyzw$
- The pencil admits an action of $G = (\mathbb{Z}/(4))^2$ (multiply coordinates by 4th roots of unity)
- Resolve singularities in the quotient $X_\psi/G$ to obtain $Y_\psi$
- $Y_\psi$ is the mirror family to smooth quartics in $\mathbb{P}^3$
- Smooth quartics in $\mathbb{P}^3$ have many complex deformation parameters; $Y_\psi$ has 1

# The Fermat quartic pencil

Let $X_\psi$ be the Fermat quartic pencil. Xenia de la Ossa and Shabnam Kadir (building on results of Dwork) showed:

$$P_{X_\psi}(T) = R_\psi(T)Q^3(T)S^6(T)$$

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where (with choices of $\pm$ depending on $p$ and $\psi$)

- $R_\psi(T) = (1 \pm pT)(1 - a_\psi T + p^2 T)$
- $Q(T) = (1 \pm pT)(1 \pm pT)$
- $S(T) = (1 - pT)(1 + pT)$ when $p \equiv 3 \bmod 4$
  $(1 \pm pT)^2$ otherwise

# Mirror Quartics

Let $Y_\psi$ be the mirror family to quartics in $\mathbb{P}^3$ (constructed using Greene-Plesser and the Fermat pencil). Then de la Ossa and Kadir showed:

$$Z(Y_\psi/\mathbb{F}_p, T) = \frac{1}{(1-T)(1-pT)^{19}(1-p^2T)R_\psi(T)}.$$

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The factor $R_\psi(T)$ corresponds to periods of the holomorphic form and its derivatives, and is invariant under mirror symmetry.

# How to Generalize?

How can one generalize this arithmetic mirror phenomenon?

# Fermat Pencils

- Daqinq Wan: For the Fermat pencil $X_\psi$ in any dimension and its Greene-Plesser mirror $Y_\psi$, the unit roots match, so

$$\#X_\psi(\mathbb{F}_q) \equiv \#Y_\psi(\mathbb{F}_q) \pmod{q}.$$

# Berglund-Hübsch-Krawitz Duality

Does Berglund-Hübsch-Krawitz (BHK) mirror symmetry have arithmetic implications?

# A matrix polynomial

Consider a polynomial $F_A$ that is the sum of $n+1$ monomials in $n+1$ variables

$$F_A := \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}}.$$

We view $F_A$ as determined by an integer matrix $A = (a_{ij})$ (with rows corresponding to monomials).

# Invertible polynomials

We say $F_A$ is invertible if:

- The matrix $A$ is invertible
- There exist positive integers called weights $p_j$ so that $d := \sum_{j=0}^{n} p_j a_{ij}$ is the same constant for all $i$
- The polynomial $F_A$ has exactly one critical point, namely at the origin.

## Calabi-Yau Condition

We say an invertible polynomial $F_A$ satisfies the Calabi-Yau condition if $d = \sum_{j=0}^{n} p_j$.

# Consequences

If a polynomial is invertible and the Calabi-Yau condition is satisfied:

- The weights determine a weighted projective space $\mathbb{WP}^n(p_0, \ldots, p_n)$
- $F_A$ determines a Calabi-Yau hypersurface $X_A$ in this weighted projective space.

# Classifying Invertible Polynomials

Kreuzer and Skarke proved that any invertible polynomial $F_A$ can be written as a sum of invertible potentials, each of which must be of one of the three *atomic types*:

$$
\begin{aligned}
W_{\text{Fermat}} &:= x^a, \\
W_{\text{loop}} &:= x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}x_1, \text{ and} \\
W_{\text{chain}} &:= x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}.
\end{aligned}
$$

# A Group Action

- Let $SL(F_A) \subset (\mathbb{C}^*)^{n+1}$ be the diagonal symmetries of $F_A$ of determinant 1.
- $SL(F_A)$ is a finite abelian group, and the coordinates of each element of $SL(F_A)$ are roots of unity.
- Let $J(F_A)$ be the trivial diagonal symmetries.
- $SL(F_A)/J(F_A)$ acts nontrivially and symplectically on $X_A$ (fixes the holomorphic $n-1$-form).

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# The BHK Mirror

We start with a manifold $X_A$, corresponding to a matrix $A$.

- Take the transpose matrix $A^T$.
- Consider the polynomial $F_{A^T}$.
- Let $\widetilde{G^T} = SL(F_{A^T})/J(F_{A^T})$.
- We obtain a dual orbifold $X_{A^T}/\widetilde{G^T}$.

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- We obtain a dual orbifold $X_{A^T}/\widetilde{G^T}$.
- In general, for any subgroup $H$ of $SL(F_A)/J(F_A)$, one may define the Berglund-Hübsch-Krawitz mirror of the orbifold $X_A/H$.

# Motivating Question

If $A^T$ and $B^T$ have common properties, do $X_A$ and $X_B$ share arithmetic properties?

# Pencils

BHK duality for a polynomial $F_A$ extends naturally to the pencil of hypersurfaces described by

$$F_A - (d^T)\psi x_0 \dots x_n,$$

where $d^T = \sum q_i$ is the sum of the dual weights.

# A common factor

## Theorem (DKSSVW)

*Let $X_{A,\psi}$ and $X_{B,\psi}$ be invertible pencils of Calabi-Yau $(n-1)$-folds in $\mathbb{P}^n$. Suppose $A$ and $B$ have the same dual weights $(q_0, \ldots, q_n)$. Then for each $\psi \in \mathbb{F}_q$ such that $\gcd(q, (n+1)d^T) = 1$ and the fibers $X_{A,\psi}$ and $X_{B,\psi}$ are nondegenerate and smooth, the polynomials $P_{X_{A,\psi}}(T)$ and $P_{X_{B,\psi}}(T)$ have a common factor $R_\psi(T) \in \mathbb{Q}[T]$ with*

$$\deg R_\psi(T) \geq D(q_0, \ldots, q_n).$$

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$$\deg R_\psi(T) \geq D(q_0, \ldots, q_n).$$

Furthermore, $\deg R_\psi(T) \leq \dim_{\mathbb{C}} H^{n-1}_{\text{prim}}(X_{A,\psi}, \mathbb{C})^{SL(F_A)}$.

# Example



For the following quartic pencils in $\mathbb{P}^3$, the dual weights are $(1,1,1,1)$ and $\deg R_\psi(T) = 3$.

| Family | Equation | $SL(F_A)/J(F_A)$ |
|---|---|---|
| $F_4$ | $x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4\psi x_0x_1x_2x_3$ | $(\mathbb{Z}/4\mathbb{Z})^2$ |
| $F_2L_2$ | $x_0^4 + x_1^4 + x_2^3x_3 + x_3^3x_2 - 4\psi x_0x_1x_2x_3$ | $\mathbb{Z}/8\mathbb{Z}$ |
| $F_1L_3$ | $x_0^4 + x_1^3x_2 + x_2^3x_3 + x_3^3x_1 - 4\psi x_0x_1x_2x_3$ | $\mathbb{Z}/7\mathbb{Z}$ |
| $L_2L_2$ | $x_0^3x_1 + x_1^3x_0 + x_2^3x_3 + x_3^3x_2 - 4\psi x_0x_1x_2x_3$ | $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ |
| $L_4$ | $x_0^3x_1 + x_1^3x_2 + x_2^3x_3 + x_3^3x_0 - 4\psi x_0x_1x_2x_3$ | $\mathbb{Z}/5\mathbb{Z}$ |

# Changing fields

For $\mathbb{F}_q$ containing sufficiently many roots of unity, we have that

$$Z(X_{\diamond,\psi}/\mathbb{F}_q, T) = \frac{1}{(1-T)(1-qT)^{19}(1-q^2T)R_\psi(T)}.$$

We may say our zeta functions are potentially equal.

# Comparing to the mirror

Let $Y_\psi$ be the family of mirror quartics we constructed earlier. Then $Z(X_{\diamond,\psi})$ and $Z(Y_\psi)$ are potentially equal for any $\diamond$.

# Example

## ♣ family

For the following quartic pencils in $\mathbb{P}^3$, the dual weights are $(4, 2, 3, 3)$ and $\deg R_\psi(T) = 6$.

| Family | Equation | $SL(F_A)/J(F_A)$ |
|---|---|---|
| $C_2F_2$ | $x_0^3x_1 + x_1^4 + x_2^4 + x_3^4 - 12\psi x_0x_1x_2x_3$ | $\mathbb{Z}/4\mathbb{Z}$ |
| $C_2L_2$ | $x_0^3x_1 + x_1^4 + x_2^3x_3 + x_3^3x_2 - 12\psi x_0x_1x_2x_3$ | $\mathbb{Z}/2\mathbb{Z}$ |

# Classifying Invertible Pencils

Other invertible K3 pencils in $\mathbb{P}^3$ have unique dual weights, so the study of $\diamond$ and $\clubsuit$ completely describes the implications of our theorem for K3 surfaces.

# Example

## Dual weights $(1, \ldots, 1)$

In $\mathbb{P}^n$, if the common dual weights are $(q_0, \ldots, q_n) = (1, \ldots, 1)$, then the common factor $R_\psi(T) \in \mathbb{Q}[T]$ has $\deg R_\psi = n$.

# Why does this work?

- The Picard-Fuchs equation satisfied by the holomorphic form depends only on the dual weights, by a result of Gährs.

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- The Picard-Fuchs equation determines a subspace of $p$-adic (Dwork) cohomology stable under the action of Frobenius . . .
- And fixed by the action of $SL(F_A)$ on cohomology.

# Hypergeometric equations

Gährs showed that the Picard-Fuchs equation is a hypergeometric differential equation. To explain her result (and define $D(q_0, \ldots, q_n)$), we need notation for the parameters.

## Rational numbers

Define

$$\begin{aligned} \alpha_j &:= \frac{j}{d^T}, \quad \text{for } j = 0, \ldots, d^T - 1; \\ \beta_{ij} &:= \frac{j}{q_i}, \quad \text{for } i = 0, \ldots, n \text{ and } j = 0, \ldots, q_i - 1. \end{aligned} \tag{1}$$

# Multisets

## Multisets

$$\boldsymbol{\alpha} := \left\{\alpha_j : j = 0, \ldots, d^T - 1\right\};$$
$$\boldsymbol{\beta}_i := \{\beta_{ij} : j = 0, \ldots, q_i - 1\}, \quad \boldsymbol{\beta} = \bigcup_{i=0}^{n} \boldsymbol{\beta}_i. \tag{2}$$

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## Intersection

Take the intersection $I = S(\boldsymbol{\alpha}) \cap S(\boldsymbol{\beta})$.

# Gährs' Theorem

Let $\delta = \psi \dfrac{d}{d\psi}$.

Theorem (Gährs)

*Let $X_{A,\psi}$ be an invertible pencil of Calabi-Yau $(n-1)$-folds determined by the integer matrix $A$. Suppose $A$ has dual weights $(q_0, \ldots, q_n)$. Then:*

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- *the order of the Picard–Fuchs equation satisfied by the holomorphic period is*

$$D(q_0, \ldots, q_n) := d^T - \#I.$$

- *the Picard–Fuchs equation is given by the differential equation*

$$\left(\prod_{i=0}^{n} q_i^{q_i}\right) \psi^{d^T} \left(\prod_{\beta_{ij} \in \boldsymbol{\beta} \setminus I} (\delta + \beta_{ij} d^T)\right) - \prod_{\alpha_j \in \boldsymbol{\alpha} \setminus I} (\delta - \alpha_j d^T).$$

# Generalized hypergeometric functions

Let $A, B \in \mathbb{N}$. Recall that a **hypergeometric function** is a function on $\mathbb{C}$ of the form:

$$\begin{aligned} {}_AF_B(\alpha;\beta|z) &= {}_AF_B(\alpha_1,\ldots,\alpha_A;\beta_1,\ldots,\beta_B|z) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_A)_k}{(\beta_1)_k\cdots(\beta_B)_k k!} z^k, \end{aligned}$$

where $\alpha \in \mathbb{Q}^A$ are *numerator parameters*, $\beta \in \mathbb{Q}^B$ are *denominator parameters*, and the Pochhammer notation is defined by:

$$(x)_k = x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}.$$

# A hypergeometric Picard-Fuchs equation

One solution to Gährs' Picard-Fuchs equation is the following hypergeometric function:

$$ {}_DF_{D-1}\left( \begin{array}{c} \alpha_i \in \boldsymbol{\alpha} \smallsetminus I \\ \beta_{ij} \in \boldsymbol{\beta} \smallsetminus (I \cup \{0\}) \end{array} ; (\textstyle\prod_i q_i^{-q_i})\psi^{-d^T} \right) $$

# Hypergeometric functions and unit roots

The unit root is determined by a formal power series depending on

$$ {}_D F_{D-1}\left( \begin{array}{c} \alpha_i \in \boldsymbol{\alpha} \smallsetminus I \\ \beta_{ij} \in \boldsymbol{\beta} \smallsetminus (I \cup \{0\}) \end{array} ; (\textstyle\prod_i q_i^{-q_i}) \psi^{-d^T} \right) $$

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$$ {}_D F_{D-1}\left( \begin{matrix} \alpha_i \in \boldsymbol{\alpha} \setminus I \\ \beta_{ij} \in \boldsymbol{\beta} \setminus (I \cup \{0\}) \end{matrix} ; (\prod_i q_i^{-q_i}) \psi^{-d^T} \right) $$

This follows from results of Miyatani (when $X_{A,\psi}$ is smooth, $\psi \neq 0$) or Adolphson-Sperber (in general).

# Unit root

## Proposition (DKSSVW)

Let $F_A(x)$ and $F_B(x)$ be invertible polynomials in $n+1$ variables satisfying the Calabi–Yau condition. Suppose $A^T$ and $B^T$ have the same weights. Then for all $\psi \in \mathbb{F}_q$ and in all characteristics (including when $p \mid d^T$), either:

- the unit root of $X_{A,\psi}$ is the same as the unit root of $X_{B,\psi}$, or
- neither variety has a nontrivial unit root.

Thus, the supersingular locus is the same for both pencils.

# Counting $\pmod{q}$

## Corollary

Let $F_A(x)$ and $F_B(x)$ be invertible polynomials in $n+1$ variables satisfying the Calabi–Yau condition. Suppose $A^T$ and $B^T$ have the same weights. Then for any fixed $\psi \in \mathbb{F}_q$ and in all characteristics (including $p \mid d^T$) the $\mathbb{F}_q$-rational point counts for fibers $X_{A,\psi}$ and $X_{B,\psi}$ are congruent as follows:

$$\#X_{A,\psi} \equiv \#X_{B,\psi} \pmod{q}.$$