The Structure of Finite Algebras and the Constraint Satisfaction Problem

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The Constraint Satisfaction Problem (CSP) appears in computer science in many forms. In the most general version a CSP instance consists of variables and constraints, and the goal is to evaluate the variables so that that all the constraints are satisfied. In a version of CSP particularly amenable to theoretical approach each constraint takes a form of a pair: a tuple of variables and a list of allowed evaluations. This restriction immediately makes the problem complete for NP. However, by further restricting the constraints allowed to appear in instances, we arrive at a very general framework capturing many natural problems in computer science.

The list of allowed evaluations in a constraint can be viewed as a relation of appropriate arity. In the parametrized (a.k.a non-uniform) version of CSP the set of relations allowed to appear in a constraint is restricted and called the *language of the CSP*. The famous dichotomy conjecture of Feder and Vardi [3] postulates that every finite language defines a CSP which is NPC or solvable in a polynomial time. The classical theorem of Schaefer [6], classifying the CSPs on two-element sets, is one of the most prominent results supporting the conjecture.

For a few years now the strongest partial results supporting the dichotomy conjecture use the algebraic approach pioneered in [5, 4]. It relies on a Galois correspondence [1] pairing a CSP language with an algebra. The algebra captures the computational complexity of the associated CSP up to a LOGSPACE reduction, but the connection goes deeper. The structure of the algebra is intrinsically connected with the types of obstacles the algorithm needs to overcome in order to verify an existence of a solution.

Many structural properties of algebras are captured by *identities* i.e. universally quantified equalities of operations. For example the identity $f(x, y) \approx f(y, x)$ states that an operation f is binary and commutative. Among the first, and most important, results of the algebraic approach is the fact that an algebra satisfies identities *which cannot be satisfied by projections* (such algebras are called *Taylor* algebras) or the associated CSP is NPC.

This year two independent papers introduced two algorithms solving CSPs for all the Taylor algebras and thus confirming the dichotomy conjecture. In each case the algorithm requires strong structural properties of the algebra to function correctly. In the paper of Andrei Bulatov [2] the structure is described by a colored graph defined on the elements of an algebra; in the other publication Dmitriy Zhuk [8] relies on the more "global" properties of being an absorbing subuniverse or a center. The goal of the workshop was to study properties of Taylor algebras using the tools developed for these proofs.

1 Specifying the problem

In both proofs one of the very first steps of the analysis is to refine the Taylor algebra. This reduces the number of operations, and at the same time (via the Galois correspondence) allows new relations into the constraint language. As long as the reduction produces a Taylor algebra the transformation has no impact on the correctness of the proof.

The first obstacle to comparing the approaches is the fact that each paper performs a different reduction. More precisely, starting with the same language, each proof allows a different set of extra relations, which makes the approaches incompatible without impacting their correctness. In order to overcome this obstacle the work during the meeting was focused on so called *Taylor minimal* algebras.

Taylor minimal algebras are the algebras corresponding to maximal tractable languages i.e. languages such that each new relation is either trivially definable (primitively positively definable) in the languages or makes in NPC. The fact that every Taylor algebra can be reduced to a minimal Taylor algebra requires some proof [7], but is true.

More importantly such an algebra cannot be further reduced without loosing the property of being a Taylor algebra and thus no reductions performed by Bulatov or Zhuk can modify it. This allows to perform a direct comparison of the structural results used in both proofs (albeit in the restricted case of Taylor minimal algebras).

The goal of the meeting can be concisely described as understanding the structure of Taylor minimal algebras by means of the proofs of the CSP dichotomy conjecture.

2 Results obtained during the meeting

Currently, there are three approaches to analyzing the structure of finite algebras that are relevant to the complexity of the CSP. Two of them are used in the dichotomy proofs mentioned above: Bulatov's local underlaying structure of the algebra captured by a directed graph (on the elements of the algebra) where each edge is a *semilattice*, *affine* or *majority* edge, and Zhuk's more global structure based on so-called centers and affine fragments of algebras. The third approach has been developed by Barto and Kozik and relies on the concept of *absorbing subuniverse* i.e. a set of elements of the algebra closed with respect to all the operations of the algebra and such that for some operation f the value $f(a_1, \ldots, a_n)$ is in the set as long as at most one a_i is outside.

During the meeting we were able to connect (to some extent) the three approaches. Most the results fall in the scope of at least one of the following broad groups:

- Refining, expanding, and reproving, using different techniques, some of the existing structural results. E.g. proving through colored graphs results that previously required using absorbing subuniverses, and the other way round.
- Comparing the constructions from the three approaches in Taylor minimal algebras. In particular, it turns out that these constructions are equivalent for the absorbing subuniverses where f can be taken binary or ternary and are not equivalent in general.
- Obtaining characterizations of algebras whose directed graph has only restricted set of colors. These characterizations can be obtained in different languages including Maltsev conditions (through identities) and through the structure of absorption.

3 Further research directions

The meeting initiated a line of research devoted to developing a systematic understanding of the structure of Taylor minimal algebras. The tools used to prove the CSP dichotomy conjecture are applicable and provide a guideline. The long–reaching goal is to construct a semblance of a structural theory for Taylor minimal algebras with a hope of eventually lifting it to all finite Taylor algebras.

References

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