The cofiber $C\tau$ and Motivic Chromatic stuff

Motivic Homotopy Theory

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This gives a symmetric monoidal model category $Spc_{\mathbb{C}}$, and there is a ***realization*** *functor R by taking $\mathbb{C}$-points*

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This gives bigraded spheres $S^{n+k,n} = \left(S^{1,0}\right)^{\wedge k} \wedge \left(S^{1,1}\right)^{\wedge n}$ for $n, k \geq 0$, and thus bigraded homotopy groups, and **bigraded everything**. . . .

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The first index $S^{\mathbf{m},n}$ is the topological dimension.
The second index $S^{m,\mathbf{n}}$ is called the weight.

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- ...etc

and they all realize to their classical analogues.

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and the element $\tau \in \mathrm{Ext}^0$ survives to a map $S^{0,-1} \xrightarrow{\tau} \widehat{S^{0,0}}_2$, but does not exist before 2-completion.

Therefore, we work in the **2-completed category**, and $S^{0,0}$ means the 2-completed sphere.

The realization functor and τ

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Question

What happens when we let $\tau = 0$?

The cofiber $C\tau$ and its homotopy

Let's look at the 2-cell complex $C\tau$ that fits in the cofiber sequence

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where $\tilde{E}_2(S^0; BP)$ is a (harmless) regrading of the Adams-Novikov E_2-page for the sphere S^0, i.e., $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$.

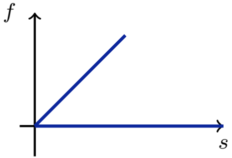
Very cool question

Question

Is there a ring structure on $C\tau$ inducing the product on $\widetilde{E}_2$-AN(S^0) ?

Notice the big vanishing regions for $C\tau$

The classical E_2-AN(S^0) has big vanishing areas:



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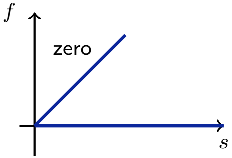

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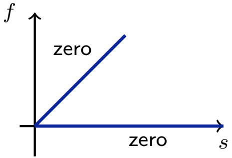

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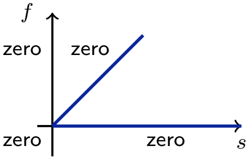

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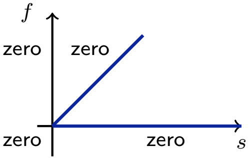

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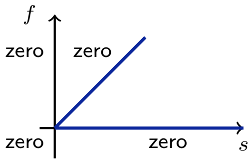

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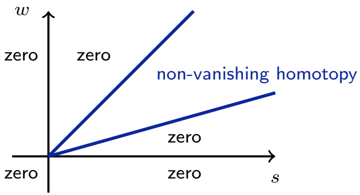

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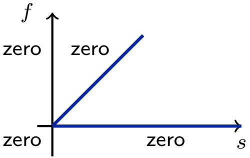

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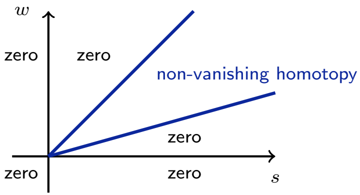

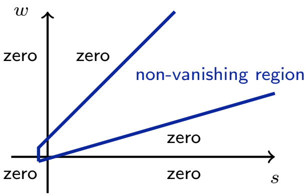

lots of vanishing in $\pi_{s,w}(C\tau)$, and lots of vanishing in $[\Sigma^{s,w}C\tau, C\tau]$.

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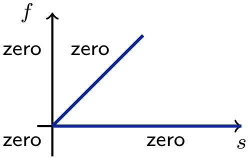

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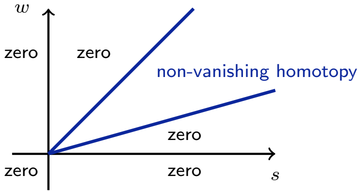

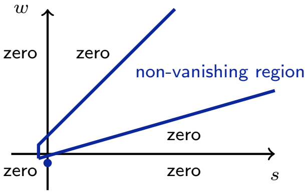

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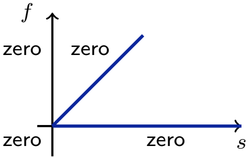

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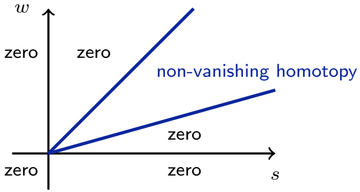

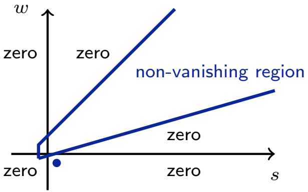

lots of vanishing in $\pi_{s,w}(C\tau)$, **and also use** $[\Sigma^{1,-1}C\tau, C\tau] = 0$.

The ring structure of $C\tau$

Smash with $- \wedge C\tau$ the defining cofiber sequence of $C\tau$

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  - μ is unique
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Theorem (G.)

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Theorem (G.)

In fact every $C\tau^n$ admits a unique E_∞-ring structure.

Operations and Co-operations on $C\tau$

Recall the maps i and p in the defining cofiber sequence of $C\tau$

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Proposition (G.)

- *The E_∞-ring spectrum $C\tau \wedge C\tau$ has homotopy ring*

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$$\pi_{*,*}(\mathrm{End}(C\tau)) \cong \widetilde{E}_2\text{-}AN(S^0)\langle x\rangle \Big/ \begin{matrix} ax - (-1)^{|a|}xa = i \circ p(a) \\ x^2 = 0 \end{matrix}$$

Applications to Motivic Chromatic Homotopy theory

Parts of the classical Chromatic story

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2. MU **detects nilpotence**, and p-locally BP does too.
3. Every $X \in \mathbf{FinSpt}_{(p)}$ has a well-defined type, and any spectrum of type n **admits a periodic self-map** inducing $\cdot v_n^k$ in $K(n)$.

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4. **No idea what to say about type.**

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Using $C\tau$, the w_i's fit in the following setting:

wBP and Morava K-theories $K(w_i)$

Theorem (G.)

1. *For every n, there is an E_∞-ring spectrum $K(w_n)$ with homotopy*

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Question

Where do the w_i's come from ?

The v_i's and the Steenrod Algebra

Voevodsky computed the motivic $H\mathbb{F}_2$-Steenrod Algebra, its dual is

$$\mathcal{A}_{*,*} \cong \mathbb{M}_2[\xi_1, \xi_2, \ldots, \tau_0, \tau_1, \ldots] \Big/ \tau_i^2 = \tau\xi_{i+1}$$

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The w_i's would like to arise from the R_i's, but they are not exterior.

Remark

- The R_i's are exterior modulo τ.
- Since $\tau\eta^4 = 0 \in \pi_{*,*}$, we need to mod out by τ if we want polynomial homotopy in the w_i's.

The w_i's from $H\mathbb{F}_2 \wedge C\tau$

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its Adams s.s. would collapse and give $\pi_{*,*}(wBP)^{\wedge}_2 \cong \mathbb{F}_2[w_0, w_1, \ldots]$.

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Corollary

There is a (almost certainly E_∞) ring spectrum wMU with homotopy

$$\pi_{*,*}(wMU) \cong \mathbb{F}_2[y_1, y_2, \ldots],$$

where $|y_i| = (4i+1, 2i+1)$, and which splits as a wedge of wBP's.

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What is the $L_{K(w_1)}S^{0,0}$?

Bonus

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1. Bonus 1: $S/2 \wedge C\tau$ admits a v_1^1-self map (instead of v_1^4 on $S/2$)
2. Bonus 2: $kO \wedge C\tau$ admits a v_1^2-self map (instead of v_1^4 on kO)

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There is no map $\Sigma^{2,1}S/2 \xrightarrow{v_1} S/2$. Indeed

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since $2 \cdot \bar{\eta}$ is not zero in $\pi_{2,1} S/2 \cong \mathbb{Z}/4$.

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After smashing with $C\tau$, there is a map $\Sigma^{2,1}C\tau/2 \xrightarrow{v_1} C\tau/2$. Indeed

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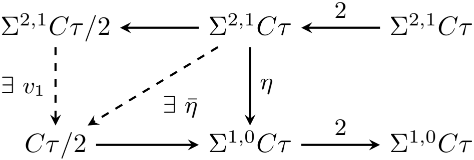

since $2 \cdot \bar{\eta}$ is zero in $\left[\Sigma^{2,1}C\tau, C\tau/2\right] \cong \mathbb{Z}/2$. More concisely, the obstruction to having a v_1^1-map is the bracket $\langle 2, \eta, 2\rangle = \tau\eta^2$, and thus $C\tau/2$ enjoys it.

Thank you for your attention !

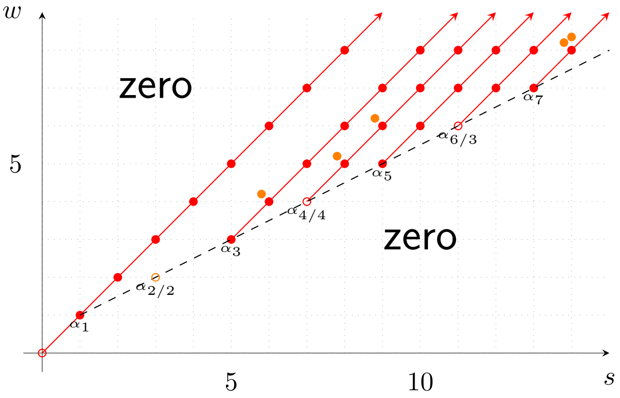

Figure: The homotopy groups $\pi_{s,w}(C\tau)$, with lots of non-nilpotent elements $2, \alpha_1, \alpha_3, \alpha_5, \alpha_7, \ldots$.