Packing Edge-disjoint Spanning Trees in Random Geometric Graphs

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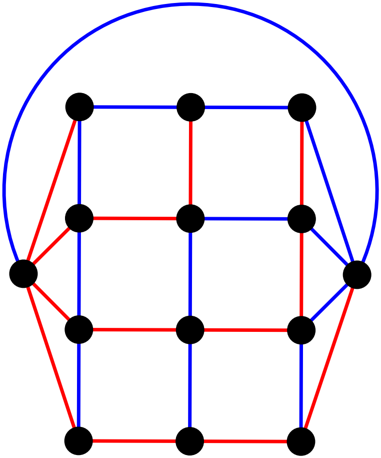
Random Geometric Graphs and Their Applications to Complex Networks

The STP number



Trivial upper bounds

- $STP(G) \le \delta(G)$;
- $STP(G) \le \lfloor m(G)/(n-1) \rfloor$.

Theorem (**G.**, Pérez-Giménez and Sato '14)

For any $0 \le m \le \binom{n}{2}$, *a.a.s.*

$$STP(G(n,m)) = \min\{\delta, m/(n-1)\},$$

where

$$\delta = \delta(G(n,m)).$$

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Previously existing gap

- Palmer and Spencer '95 – the STP number for $G(n, m)$ where $\delta = O(1)$;
- Catlin, Chen and Palmer '93 – the STP number and the arboricity when $m \approx n^{4/3}$;
- Chen, Li and Lian '13 – the STP number for $m \leq (1.1/2)n \log n$.

Tools

Theorem (Tutte '61, Nash-Williams '61)

G contains k edge-disjoint spanning trees if and only if

$$\frac{m(\mathcal{P})}{|\mathcal{P}|-1} \geq k, \quad \forall \mathcal{P}.$$

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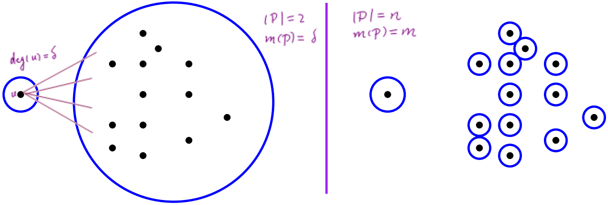

Random geometric graphs

- 1×1 torus;
- n vertices chosen uniformly from $[0, 1] \times [0, 1]$;
- two vertices adjacent if their Euclidean distance is at most r;
- $G(n; r)$.
- $STP(G(n; r))$?
- $A(G(n; r))$?

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- Small sets of vertices induce sparse subgraphs;
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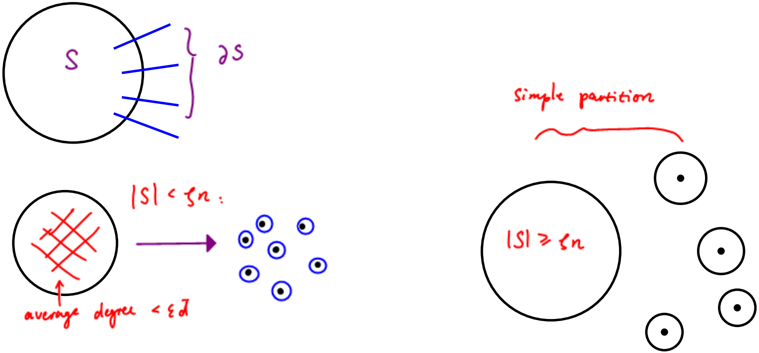

What's different about $G(n; r)$?

- Small cliques;
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Main result

Let $p(r) = \pi r^2$. Then $d_u \sim Binomial(n, p(r))$. So the degree distribution of $G(n; r)$ is almost the same as that of $G(n, p(r))$.

Theorem (G., Pérez-Giménez and Sato '16$^+$)

Assume $\epsilon_0 > 0$ is a sufficiently small constant. Then, for any r where $p(r) \le (1+\epsilon_0)\log n/n$, a.a.s.

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Connected partitions suffice

Connected partitions $\mathcal{P}$:

- Each part $S \in \mathcal{P}$ induces a connected subgraph;
- If $S_1, S_2 \in \mathcal{P}$ and $S_1 \cup S_2$ induces a clique then $|S_1||S_2| = 1$.

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Subgraph induced by light vertices

Light vertices are those with degree $\leq 6\delta$.

Lemma

- *G_L is a union of cliques; each clique is composed of a set of vertices inside a ball of radius $r/4$;*
- *Every clique in G_L has order at most 4δ;*
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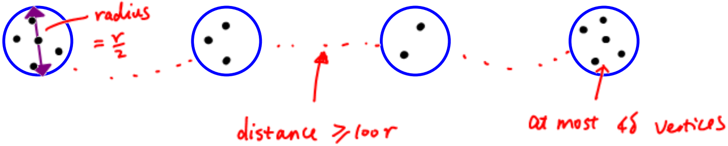

Sensitive parts

- F are parts in $\mathcal{P}$ containing only one vertex, and the vertex is light.

Necessary conditions for $T(G) = \delta$

- No adjacent vertices both with minimum degree;
- No sets S with $|\partial S| < \delta$;
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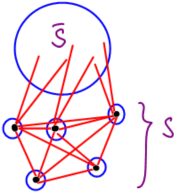

$$|P| = |S| + 1$$

$$m(P) = \bar{d}_S \cdot |S| - \frac{|S|(|S|-1)}{2}$$

$$< \left(\delta + \frac{|S|-1}{2}\right)|S| - \frac{|S| \cdot (|S|-1)}{2}$$

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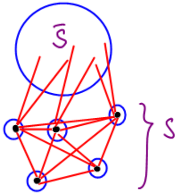

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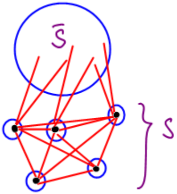

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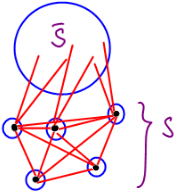

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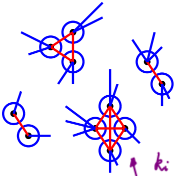

$$|\partial F| = \sum_{i=1}^{h} \left(\bar{d}_i k_i - \binom{k_i}{2} \right) \quad \bar{d}_i \geq \delta + (k_i - 1)/2$$

$$\geq \sum_{i=1}^{h} \delta k_i = \delta |F|$$

$$m(\mathcal{P}) = |\partial F| + \frac{1}{2} \sum_{S \notin F} |\partial S \setminus \partial F| \geq |\partial F| + \frac{1}{2}(|\mathcal{P}| - |F| - 1) \cdot 2\delta = \delta(|\mathcal{P}| - 1)$$

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k_i vertices

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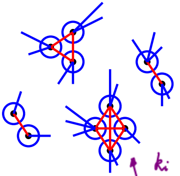

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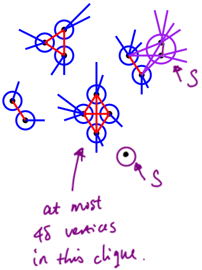
$|\partial S \setminus \partial F| \geq 2\delta$



① if $|S|=1$. v is not light. $\Rightarrow \deg(v) \geq 6\delta$.

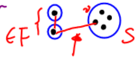
$|\partial S \setminus \partial F| \geq 6\delta - 4\delta = 2\delta$.

② diameter $(S) < \alpha r$.

We can show $|\partial S| \geq 2\delta$.

Either $\partial S \cap \partial F = \phi$. $\rightarrow$ done.

or



$\geq \frac{r}{2}$ apart. $\Rightarrow \deg(v) \geq \gamma \log n$.

α was chosen so that $|B_{\alpha r}(v)| < \frac{\gamma}{2}\log n$. $\Rightarrow |S| < \frac{\gamma}{2}\log n$

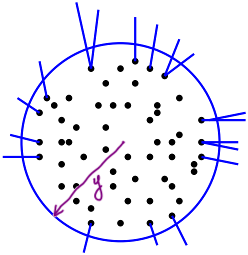
Then,

$|\partial S \setminus \partial F| \geq \deg(v) - |S| - 4\delta \geq \frac{\gamma}{2}\log n - 4\delta \geq 2\delta$.

③ diameter $(S) \geq \alpha r$

...

When $p(r) = \Omega(\log n/n)$.



S

$$|E(S)| = \frac{1}{2}\left(\bar{d}_S \cdot |S| - |\partial S|\right)$$

$\mathcal{P} \to \mathcal{P}'$ by splitting S.

$$\frac{m(\mathcal{P}')}{|\mathcal{P}'|-1} = \frac{m(\mathcal{P}) + |E(S)|}{|\mathcal{P}|-1 + (|S|-1)}$$

Is $\dfrac{|E(S)|}{|S|-1} \le \dfrac{m}{n-1}$?

$$\frac{m}{n-1} \approx \frac{1}{2}\bar{d}. \qquad |S| \approx \pi y^2 \cdot n$$

$$|\partial S| = \Theta(2\pi y \cdot r \cdot n \cdot \bar{d})$$

If $d(S) = \bar{d} \cdot |S| + O_p(\sqrt{\bar{d} \cdot |S|})$

$$\bar{d} \approx \pi r^2 n$$

Then $$\frac{|E(S)|}{|S|-1} \approx \frac{1}{2}\left(\frac{\bar{d}\,|S| + O_p(\sqrt{\bar{d}|S|}) - \Theta(yrn\bar{d})}{|S|}\right)$$

$$= \frac{1}{2}\left(\bar{d} + O_p\left(\frac{r}{y}\right) - \Theta\left(\frac{r^3 n}{y}\right)\right)$$

We have $\frac{r}{y} \ll \frac{r^3 n}{y}$, as $r^2 n = \Omega(\log n)$.

Open problems

- Concentration of $\bar{d}(S)$?
- $STP(G(n;r)) = \lfloor m/(n-1) \rfloor$, for $r \geq C\sqrt{\log n/n}$?
- $STP(G(n;r)) = \min\{\delta, \lfloor m/(n-1) \rfloor\}$ for any r?
- $A(G(n;r))$?