

On the maximum of the characteristic polynomial of the Circular Beta Ensemble

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Presentation of the setting

- ▶ We consider the Circular Beta Ensemble ($C\beta E$), corresponding to n points on the unit circle \mathbb{U} , whose probability density with respect to the uniform measure on \mathbb{U}^n is given by

$$C_{n,\beta} \prod_{1 \leq j < k \leq n} |\lambda_j - \lambda_k|^\beta,$$

for some $\beta > 0$.

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for some $\beta > 0$.

- ▶ For $\beta = 2$, one gets the distribution of the eigenvalues of a Haar-distributed matrix on the unitary group $U(n)$. Other matrix models has been found by Killip and Nenciu in 2004 for general β .

- ▶ If $(\lambda_j^{-1})_{1 \leq j \leq n}$ are the eigenvalues of a random matrix, one can consider the characteristic polynomial:

$$X_n(z) = \prod_{j=1}^n (1 - \lambda_j z),$$

and its logarithm

$$\log X_n(z) = \sum_{j=1}^n \log(1 - \lambda_j z),$$

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which can be well-defined in a continuous way, except on the half-lines $\lambda_j^{-1} [1, \infty)$.

- ▶ We will be interested in the extremal values of $\log X_n(z)$ on the unit circle.

- ▶ It can be proven that $\left(\sqrt{\beta/2} \log X_n(z)\right)_{z \in \mathbb{D}}$ (\mathbb{D} being the open unit disc) tends in distribution to a complex Gaussian holomorphic function: for $\beta = 2$, it is a direct consequence of a result by Diaconis and Shahshahani (1994) on the moments of the traces of the CUE.

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- ▶ This Gaussian function \mathbb{G} has the following covariance structure:

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- ▶ The variance of \mathbb{G} goes to infinity when $|z| \rightarrow 1$, and for $z \in \mathbb{U}$, $\log X_n(z)$ does not converge in distribution.

- ▶ When n goes to infinity,

$$\sqrt{\frac{\beta}{2 \log n}} \log X_n(z) \xrightarrow{n \rightarrow \infty} \mathcal{N}^{\mathbb{C}},$$

where $\mathcal{N}^{\mathbb{C}}$ denotes a complex Gaussian variable Z such that

$$\mathbb{E}[Z] = \mathbb{E}[Z^2] = 0, \quad \mathbb{E}[|Z|^2] = 1.$$

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- ▶ Without normalization, $(\sqrt{\beta/2} \log X_n(z))_{z \in \mathbb{C}}$ tends in distribution to a complex Gaussian field on the unit circle, whose correlation between points $z, z' \in \mathbb{U}$ is given by $\log |z - z'|$. Note that this field is not defined on single points, since the correlation has a logarithmic singularity when z' goes to z .

Presentation of the setting

Statement of the main result

Orthogonal polynomial on the unit circle

Sketch of proof of a non-sharp upper bound

Sketch of proof of a sharper upper bound

Strategy for a lower bound

- ▶ The logarithm of the characteristic polynomial, multiplied by $\sqrt{\beta/2}$, is a rather complex (yet integrable) regularization of the log-correlated Gaussian field given above.

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- ▶ In this regularization, the correlation of the field saturates when $|z - z'|$ is of order $1/n$, which is consistent with the result by Keating and Snaith.
- ▶ For this kind of regularization, it is conjectured that the maximum of the field is of order $\log n - (3/4) \log \log n$. This behavior (in particular the constant $-3/4$) is believed to be universal, i.e. not depending on the detail of the model.

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- ▶ For this kind of regularization, it is conjectured that the maximum of the field is of order $\log n - (3/4) \log \log n$. This behavior (in particular the constant $-3/4$) is believed to be universal, i.e. not depending on the detail of the model.
- ▶ Such result has been proven for Gaussian regularizations (by Madaule, in 2015, then generalized by Ding, Roy and Zeitouni), for branching random walks and branching Brownian motion.

- ▶ From the log-correlated field, one can also define the *Gaussian multiplicative chaos*, introduced by Kahane in 1985, as a random measure $\mu^{(\alpha)}$, formally given by

$$\frac{d\mu^{(\alpha)}}{d\mu}(z) = \frac{e^{\alpha G_{\mathbb{U}}(z)}}{\mathbb{E}[e^{\alpha G_{\mathbb{U}}(z)}]}$$

where $G_{\mathbb{U}}$ is a log-correlated Gaussian field on \mathbb{U} , and μ is the uniform measure on \mathbb{U} .

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- Webb (2014) has proven that for $\beta = 2$ and $\alpha < \sqrt{2}$, and for $\mu_{X_n}^{(\alpha)}$ given by

$$\frac{d\mu_{X_n}^{(\alpha)}}{d\mu}(z) = \frac{|X_n(z)|^\alpha}{\mathbb{E}[|X_n(z)|^\alpha]},$$

one has

$$\mu_{X_n}^{(\alpha)} \xrightarrow{n \rightarrow \infty} \mu^{(\alpha)}.$$

Statement of the main result

- ▶ For $\beta = 2$, Fyodorov, Hiary and Keating (2012), have given a conjecture on the maximum of the characteristic polynomial, which is the following:

$$\sup_{z \in \mathbb{U}} \log |X_n(z)| - \left(\log n - \frac{3}{4} \log \log n \right) \xrightarrow{n \rightarrow \infty} \frac{1}{2} (K_1 + K_2),$$

in distribution, where K_1 and K_2 are two independent Gumbel random variables.

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in distribution, where K_1 and K_2 are two independent Gumbel random variables.

- ▶ In November 2015, Arguin, Belius and Bourgade have proven that

$$\frac{\sup_{z \in \mathbb{U}} \log |X_n(z)|}{\log n} \xrightarrow{n \rightarrow \infty} 1$$

in probability.

- ▶ In February 2016, Paquette and Zeitouni have proven:

$$\frac{\sup_{z \in \mathbb{U}} \log |X_n(z)| - \log n}{\log \log n} \xrightarrow{n \rightarrow \infty} -\frac{3}{4}.$$

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- ▶ We expect that the conjecture of Fyodorov, Hiary and Keating can be generalized to β ensembles:

$$\sqrt{\beta/2} \sup_{z \in \mathbb{U}} \log |X_n(z)| - \left(\log n - \frac{3}{4} \log \log n \right) \xrightarrow{n \rightarrow \infty} K,$$

where K is a limiting random variable. It may be possible that $2K$ is the sum two independent Gumbel variables, but we have no argument supporting such a statement.

Such a result seems very challenging. However, in a work in progress, we expect to be able to prove the following result:

Conjecture

For any function h tending to infinity at infinity,

$$\left| \sqrt{\beta/2} \sup_{z \in \mathbb{U}} \Re \log X_n(z) - \left(\log n - \frac{3}{4} \log \log n \right) \right| \leq h(n),$$

$$\left| \sqrt{\beta/2} \sup_{z \in \mathbb{U}} \Im \log X_n(z) - \left(\log n - \frac{3}{4} \log \log n \right) \right| \leq h(n),$$

with probability tending to 1 when n goes to infinity.

The statement on the imaginary part gives information on the number of eigenvalues lying on arcs of the unit circle.

If the result above is true, we have the following:

Corollary

For $z_1, z_2 \in \mathbb{U}$, let $N(z_1, z_2)$ be the number of points λ_j lying on the arc coming counterclockwise from z_1 to z_2 , and $N_0(z_1, z_2)$ its expectation (i.e. the length of the arc multiplied by $n/2\pi$). Then,

$$\left| \pi \sqrt{\beta/8} \sup_{z_1, z_2 \in \mathbb{U}} |N(z_1, z_2) - N_0(z_1, z_2)| - \left(\log n - \frac{3}{4} \log \log n \right) \right| \leq h(n)$$

with probability tending to 1 when n goes to infinity.

- ▶ In the sequel of the talk, we will sketch a proof of the following result we have completely checked:

Theorem

With probability tending to 1,

$$\sqrt{\beta/2} \sup_{z \in \mathbb{U}} \Re \log X_n(z) \leq \log n - \frac{3}{4} \log \log n + \frac{3}{2} \log \log \log n + h(n),$$

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- ▶ At the end of the talk, we will briefly give some elements of the proof of the lower bound part of the stronger result stated above.

Orthogonal polynomials on the unit circle

If ν is a probability measure on the unit circle, the Gram-Schmidt procedure applied on $L^2(\nu)$ to the sequence $(z_k)_{k \geq 0}$ gives a sequence $(\Phi_k)_{0 \leq k < m}$ of monic orthogonal polynomials, m being the (finite or infinite) cardinality of the support of ν . If $m < \infty$, the procedure stops after Φ_{m-1} since all $L^2(\nu)$ is spanned: we then define

$$\Phi_m(z) := \prod_{\lambda \in \text{Supp}(\nu)} (z - \lambda),$$

which vanishes in $L^2(\nu)$. Moreover, we define $\Phi_k^*(z) := z^k \overline{\Phi_k(1/\bar{z})}$.

- ▶ There exists a sequence $(\alpha_j)_{0 \leq j < m}$ of complex numbers, $|\alpha_j| = 1$ if $j = m - 1 < \infty$, $|\alpha_j| < 1$ otherwise, called *Verblunsky coefficients*, such that the polynomials above satisfy the so-called *Szegő recursion*: for $j < m$,

$$\Phi_{j+1}(z) = z\Phi_j(z) - \bar{\alpha}_j\Phi_j^*(z),$$

$$\Phi_{j+1}^*(z) = -\alpha_j z\Phi_j(z) + \Phi_j^*(z).$$

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- ▶ Moreover, Killip and Nenciu have found an explicit probability distribution for the Verblunsky coefficients, for which one can recover the characteristic polynomial of the Circular Beta Ensemble.

- ▶ Let $(\alpha_j)_{j \geq 0}$, η be independent complex random variables, rotationally invariant, such that $|\alpha_j|^2$ is Beta($1, (\beta/2)(j+1)$)-distributed and $|\eta| = 1$ a.s.

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- ▶ Then, we have the equality in distribution:

$$X_n(z) = \Phi_{n-1}^*(z) - z\eta\Phi_{n-1}(z).$$

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- ▶ If we couple the polynomials in such a way that we have actually an equality, then

$$\left(\sup_{z \in \mathbb{U}} |\log X_n(z) - \log \Phi_{n-1}^*(z)| \right)_{n \geq 1}$$

is tight: it is then sufficient to study the extreme values of $\log \Phi_n^*$ instead of $\log X_n$.

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- ▶ We have, for $\theta \in [0, 2\pi)$,

$$\log \Phi_k^*(e^{i\theta}) = \sum_{j=0}^{k-1} \log \left(1 - \gamma_j e^{i\psi_j(\theta)} \right).$$

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- ▶ The so-called *relative Prüfer phases* $(\psi_k)_{k \geq 0}$ satisfy:

$$\psi_k(\theta) = (k+1)\theta - 2 \sum_{j=0}^{k-1} \log \left(\frac{1 - \gamma_j e^{i\psi_j(\theta)}}{1 - \gamma_j} \right).$$

Sketch of proof of a non-sharp upper bound

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- ▶ Indeed, if \mathbb{U}_m denotes the set of m -th roots of unity, we have for all polynomials Q of degree at most n :

$$\sup_{z \in \mathbb{U}} |Q(z)| \leq 14 \sup_{z \in \mathbb{U}_{2n}} |Q(z)|.$$

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- ▶ If $Q(0) = 1$ and Q has all roots outside the unit disc, then

$$\sup_{z \in \mathbb{U}} \text{Arg}(Q(z)) \leq \sup_{z \in \mathbb{U}_n} \text{Arg}(Q(z)) + 2\pi.$$

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- ▶ By computing and then estimating the exponential moments of this sum of independent random variables, we get for $s > 0, t \in \mathbb{R}$,

$$\mathbb{E}[e^{s\Re \log \Phi_k^*(z) + t\Im \log \Phi_k^*(z)}] \leq (ke)^{(s^2 + t^2)/(2\beta)}.$$

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- ▶ Using a Chernoff bound with $s = \sqrt{2\beta}$, $t = 0$, we deduce that for $n \rightarrow \infty$,

$$\mathbb{P}\left(\sqrt{\frac{\beta}{2}} \Re \log \Phi_n^*(z) \geq \log n + h(n)\right) = o(1/n)$$

and the same for the imaginary part.

- ▶ Using a union bound on the $2n$ -th roots of unity,

$$\mathbb{P} \left(\sqrt{\frac{\beta}{2}} \sup_{z \in \mathbb{U}} \Re \log \Phi_n^*(z) \leq \log n + h(n) \right) \xrightarrow{n \rightarrow \infty} 1,$$

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- ▶ Moreover, if we define

$$\mathcal{B}_n := \{ \lfloor e^j \rfloor, 0 \leq j \leq \lfloor \log n \rfloor \} \cup \{n\},$$

then

$$\mathbb{P} \left(\forall k \in \mathcal{B}_n, \sup_{z \in \mathbb{U}} \Re \log \Phi_k^*(z) \leq \log k + \log \log n + h(n) \right) \xrightarrow{n \rightarrow \infty} 1.$$

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- ▶ This estimate is useful in order to prove a sharper upper bound.

Sketch of proof of a sharper upper bound

- ▶ In order to show the sharper upper bound previously stated, it is sufficient to show:

$$\mathbb{P} \left(\forall k \in \mathcal{B}_n, \sup_{z \in \mathbb{U}} \Re \log \Phi_k^*(z) \leq \log k + \log \log n + h(n), \right.$$

$$\left. \sup_{z \in \mathbb{U}} \Re \log \Phi_n^*(z) \geq \log n - \frac{3}{4} \log \log n + \frac{3}{2} \log \log \log n + h(n) \right) \xrightarrow[n \rightarrow \infty]{} 0.$$

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- ▶ By doing a union bound on \mathbb{U}_{2n} , it is sufficient to prove that the probability of the same event for a single $z \in \mathbb{U}$ is $o(1/n)$ when n goes to infinity.

- ▶ For fixed $z \in \mathbb{U}$, $(\log \Phi_k^*(z))_{k \geq 0}$ is a random walk with independent increments, given by $\log(1 - \gamma_k)$.

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- ▶ We have an equality in law:

$$\log(1 - \gamma_k) = \log \left(1 - e^{i\Theta_k} \sqrt{\frac{E_k}{E_k + \Gamma_k}} \right)$$

where $(E_k)_{k \geq 0}$, $(\Gamma_k)_{k \geq 0}$, $(\Theta_k)_{k \geq 0}$ are independent variables, respectively exponentially distributed, Gamma of parameter $(\beta/2)(k + 1)$ and uniform on $[0, 2\pi)$.

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- ▶ If we replace $E_k + \Gamma_k$ by $(\beta(k+1))/2$ and $\log(1 - y)$ by $-y$, we get a Gaussian variable of variance $1/(\beta(k+1))$.

- ▶ One can prove that $(\sqrt{\beta/2} \log \Phi_k^*(z))_{k \geq 0, z \in \mathbb{U}}$ can be coupled, with an a.s. bounded difference, with a field $(Z_k(z))_{k \geq 0, z \in \mathbb{U}}$, with complex Gaussian marginals, with independent increments for fixed θ :

$$Z_k(e^{i\theta}) := \sum_{j=0}^{k-1} \frac{\mathcal{N}_j^{\mathbb{C}} e^{i\psi_j(\theta)}}{\sqrt{j+1}}.$$

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- ▶ In this way, we can deduce that it is essentially sufficient to show (N corresponding to $\log n$), for a Brownian motion W that:

$$\mathbb{P}\left(\forall j \in \{1, 2, \dots, N-1\}, W_j \leq \sqrt{2}(j + \log N + h(N)),\right)$$

$$W_N \geq \sqrt{2}\left(N - \frac{3}{4} \log N + \frac{3}{2} \log \log N + h(N)\right) = o(e^{-N}).$$

- ▶ Using Girsanov's theorem, it is enough to show

$$\mathbb{P} \left(\forall j \in \{1, 2, \dots, N-1\}, W_j \leq \sqrt{2} (\log N + h(N)), \right.$$

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- ▶ This result can be deduced from a suitable version of the ballot theorem, or from the joint law of a Brownian motion and its past supremum.
- ▶ We expect that one can remove the term $(3/2) \log \log \log n$ in the main result by suitable optimizing the barrier $\log k + \log \log n + h(n)$ in the proof. This would give a sharp upper bound (i.e. with a tight difference with the conjectured behavior).

Strategy for a lower bound

- ▶ In order to get a sharp lower bound, we would have to show that with high probability, there exists $\theta \in [0, 2\pi)$ such that

$$\Re Z_n(e^{i\theta}) \geq \log n - \frac{3}{4} \log \log n - h(n).$$

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- ▶ Let $E_n(\theta)$ be any event implying the previous inequality. It is sufficient to show:

$$\mathbb{P}(N_n > 0) \xrightarrow[n \rightarrow \infty]{} 1,$$

where N_n is the number of $j \in \{0, \dots, n-1\}$ such that $E_n(e^{2i\pi j/n})$ occurs.

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- ▶ For that, we need to choose events $E_n(\theta)$, in such a way that their probability is not too small and that $E_n(\theta)$ and $E_n(\theta')$ are not too much correlated if θ is not too close to θ .

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- ▶ We can then do similar computations as for branching Gaussian random walks.

Thank you for your attention!