Towards the continuous analogue of Kazhdan-Lusztig correspondence

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(Semi) Physical Motivation

Chern-Simons theory
for compact group G (say, $G = SU(2)$)

Correlation functions
of Wilson loops

boundary

Knot invariants
(e.g. Jones polynomials)

WZW model

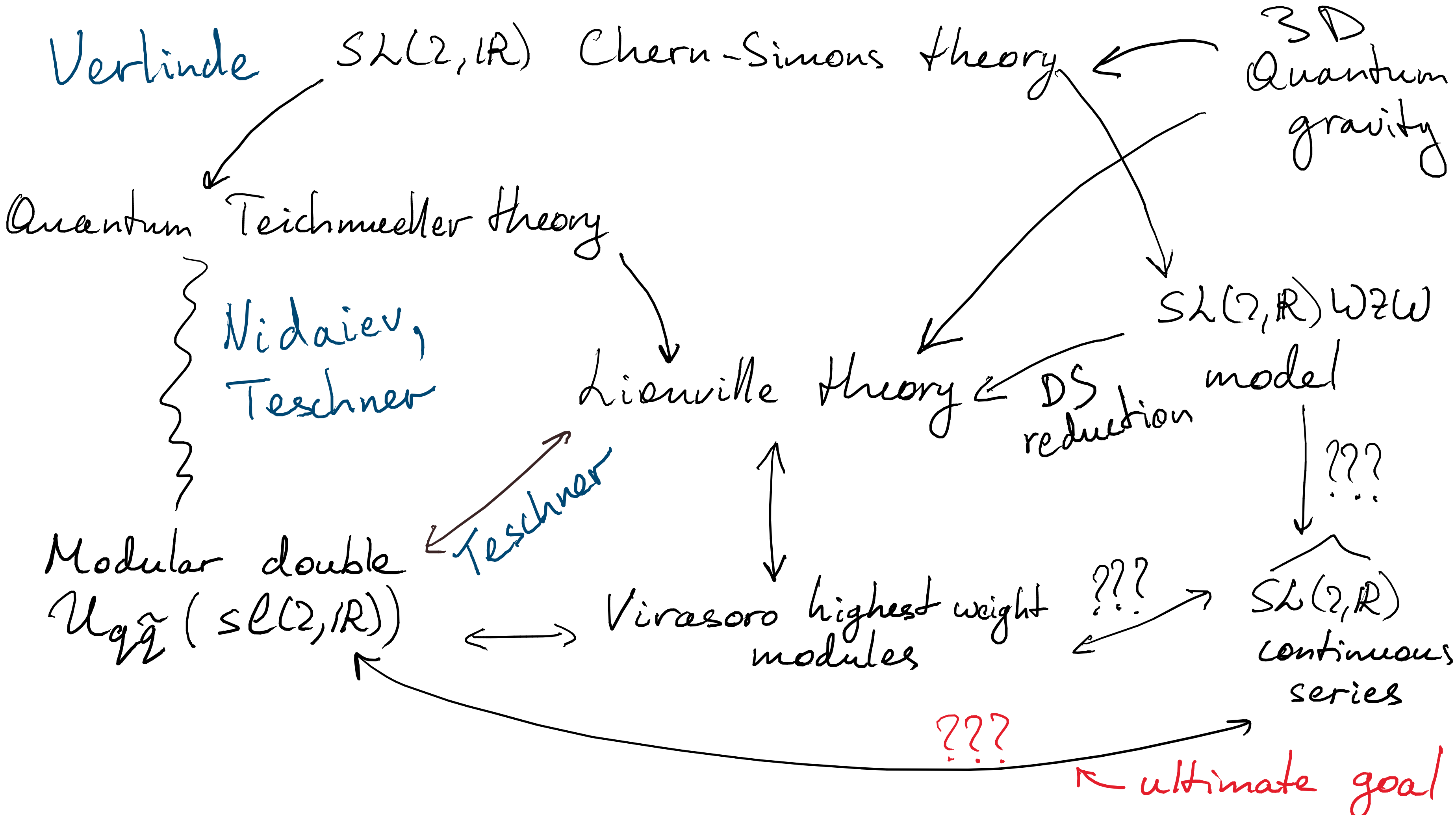
Moore-Seiberg

Finite-dimensional
irreps of $U_q(\mathfrak{g})$

Kazhdan-Lusztig

Highest weight
modules of the
affine algebra $\hat{\mathfrak{g}}$

W-algebra
(Virasoro)



In this talk: Construction of the analogue of the continuous series for $sl(2,\mathbb{R})$.

Based on: I. B. Frenkel, A. M. Zeitlin, CMP 326 (2014)

A. M. Zeitlin, JFA 263 (2012)

A. M. Zeitlin, arXiv: 1509.06072

Modular double representations of $\mathcal{U}_{q\tilde{q}}(sl(2,\mathbb{R}))$

Let $q = e^{2i\pi b^2}$, $\tilde{q} = e^{2i\pi b^{-2}}$, $0 < b^2 < 1$

$\mathcal{U}, V$ are unbounded self-adjoint operators on $L^2(\mathbb{R})$ defined by the formulas

$$\mathcal{U} = e^{2\pi b x} \qquad V = e^{2\pi b p} \qquad [p, x] = \frac{1}{2\pi i}$$

on $\mathcal{W} = \{ e^{-\alpha x^2 + \beta x} P(x), \ \mathrm{Re}\,\alpha > 0 \}$

$$b \to b^{-1} \qquad \mathcal{U} \to \tilde{\mathcal{U}}, \ V \to \tilde{V}$$

$$\mathcal{U}V = q^2 V\mathcal{U}, \qquad \tilde{\mathcal{U}}\tilde{V} = \tilde{q}^2 \tilde{V}\tilde{\mathcal{U}}$$

$\mathcal{U}_q(sl(2,\mathbb{R}))$: $\quad E = i\,\dfrac{V + \mathcal{U}^{-1} Z}{q - q^{-1}}, \quad F = i\,\dfrac{\mathcal{U} + V^{-1} Z}{q - q^{-1}}$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} \qquad K = q^{-1}\mathcal{U}V$$

Modular double $\mathcal{U}_{q\tilde{q}}(s\ell(2,\mathbb{R}))$: "commuting" families $\to$ E, F, K ; $\tilde{E}, \tilde{F}, \tilde{K}$ $\quad b \to b^{-1}$

Some "magic" formulas:

$$(u+v)^{1/b^2} = u^{1/b^2} + v^{1/b^2}, \quad e^{1/b^2} = \tilde{e} \quad \text{etc.}$$

(here $e = (2\sin\pi b^2)E$)

One can show that z, z^{-1} representations are equivalent

Denoting $P_\alpha \cong L^2(\mathbb{R})$, so that $\alpha = \log z$, one observes that:

$$P_{\alpha_2} \otimes P_{\alpha_1} \cong \int^{\oplus} d\alpha_3 \, P_{\alpha_3},$$ (Ponsot, Teschner, 2000)

so that the corresponding $3j$ symbol:

$C(\alpha_3 | \alpha_2, \alpha_1)$:

ratio of products of quantum dilog functions

$$f(x_2, x_1) \longmapsto F(f)(\alpha_3, x_3) = \int_{\mathbb{R}} dx_2 \, dx_1 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} f(x_2, x_1)$$

Highest weight representations of $\hat{\mathfrak{g}}$ and braided tensor structure

$\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}c \qquad a \otimes t^n = a_n \ , \ a \in \mathfrak{g}$

$$[a_n, b_m] = [a,b]_{n+m} + \langle a,b \rangle c\, m\, \delta_{m+n,0}$$

Highest weight modules:

$\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus d \qquad$ Let L_λ be a highest weight module for $\mathfrak{g}$

λ - dominant integral weight

$V_{\lambda,k} = \mathrm{Ind}_{\tilde{\mathfrak{g}}^+}^{\tilde{\mathfrak{g}}} L_\lambda$ where d acts on L_λ as $-\Delta(\lambda) = \frac{\langle \lambda, \lambda+2\rho \rangle}{2k+h^\vee}$

$\tilde{\mathfrak{g}}^+ = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c \oplus \mathbb{C}d \subset \tilde{\mathfrak{g}} \qquad$ (c acts as k)

Braided tensor category structure:

$$a_n \, \Phi^{\nu}_{\lambda\mu}(z) = \Phi^{\nu}_{\lambda\mu}(z)\, \Delta_{z,0}(a_n)$$

$$\Phi^{\nu}_{\lambda\mu}(z) : V_{\lambda,k} \otimes V_{\mu,k} \longrightarrow V_{\nu,k}[[z,z^{-1}]]\, z^{\Delta_\nu - \Delta_\mu - \Delta_\lambda}$$

In the equivalent braided tensor category for $U_q(\mathfrak{g})$, $q = e^{\frac{\pi i}{k+h^\vee}}$

Correlators and Frenkel-Zhu formula

$$a \longrightarrow a(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n-1}, \qquad a \in \mathfrak{g} \qquad\qquad \mathrm{tr} : \mathfrak{g} \times \cdots \times \mathfrak{g} \longrightarrow \mathbb{C}$$

correlator of currents

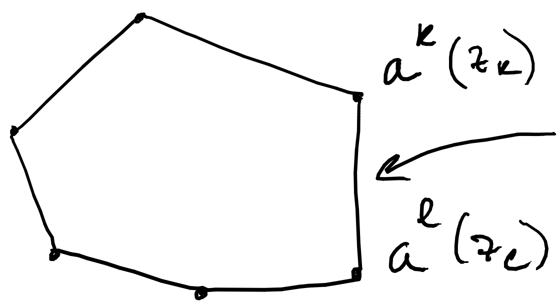
$$\langle v', a^1(z_1) \cdots a^n(z_n) v \rangle =$$

$$|z_1| > \dots |z_n| > 0$$

$$= \sum_{\text{partitions}} \frac{\mathrm{tr}(a^{1,1} \cdots a^{1,j_1})}{(z_{1,1}-z_{1,2}) \cdots (z_{1,j_1}-z_{1,1})} \cdots\cdots \frac{\mathrm{tr}(a^{r,1} \cdots a^{r,j_r})}{(z_{r,1}-z_{r,2}) \cdots (z_{r,j_r}-z_{r,1})} \cdot \frac{\langle v', a_{i_1} \cdots a_{i_m} v \rangle \quad (-k)^r}{(z_{i_1}-z_{i_2}) \cdots (z_{i_{m-1}}-z_{i_m}) z_{i_m}}$$

number of cycles in the partition

Cycle:



$$\frac{1}{z_k - z_l}$$

This is a motivational example for us: correlator, described by Feynman-type graphs determine the bilinear form

Construction of the continuous series

i) Start from $\widehat{ax+b}$ algebra and its representations

ii) Construct "regularized" currents of $\widehat{sl}(2,\mathbb{R})$ "acting" on $F_p \otimes V$

Fock module for Heisenberg algebra

representation of $\widehat{ax+b}$

iii) True correlators diverge! We found a method to describe regularized correlators via Feynman-like diagrams and to eliminate divergent graphs preserving algebraic structure.

$sl(2,\mathbb{R})$ via $*$-algebras $\mathcal{A}, \mathcal{K}$

$\mathcal{A}$: $[h, e^{\pm}] = \pm i e^{\pm}$, $e^{\pm} e^{\mp} = 1$, $h^* = h$, $e^{\pm *} = e^{\pm}$

$\mathcal{K}$: $[h, \alpha^{\pm}] = \mp \alpha^{\pm}$, $\alpha^{\pm}\alpha^{\mp} = 1$, $h^* = h$, $\alpha^{\pm *} = \alpha^{\mp}$

Representation: $h = i\frac{d}{dx}$ $\quad e^{\pm} = e^{\pm x}$ on $\mathcal{L}^2(\mathbb{R})$

$h = i\frac{d}{d\phi}$ $\quad \alpha^{\pm} = e^{\pm i\phi}$ on $\mathcal{L}^2(S^1)$

Continuous series of $sl(2,\mathbb{R})$ via $\mathcal{A}$ and $\mathcal{K}$ algebras:

$sl(2,\mathbb{R})$: $[E,F] = H$, $[H,E] = 2E$, $[H,F] = -2F$, $E = -E^*$, $F = -F^*$, $H = -H^*$

$su(1,1)$: $[J^3, J^{\pm}] = \pm 2i J^{\pm}$, $[J^+, J^-] = -iJ^3$, $J^{+*} = -J^-$, $J^{3*} = -J^3$

Using $\mathcal{K}$:

$$J^{\pm} = \frac{i}{2}(\alpha^{\pm} h + h\alpha^{\pm}) \mp \lambda \alpha^{\pm}, \quad \lambda \in \mathbb{R}$$

$$J^3 = 2ih$$

Similarly for $\mathcal{A}$ (exercise). $\qquad$ Relation: $J^3 = E + F$

$$J^{\pm} = E - F \mp iH$$

Loop version:

$$h_n, \alpha^\pm_n \quad n \in \mathbb{Z} \quad h(u) = \sum_n h_{-n} e^{inu}, \quad \alpha^\pm(u) = \sum_n \alpha^\pm_{-n} e^{inu}$$

$$[h(u), \alpha^\pm(v)] = \mp \alpha^\pm(v)\,\delta(u-v), \quad \alpha^+(u)\,\alpha^-(u) = 1,$$

$$h(u) = h(u)^*, \quad \alpha^{\pm *}(u) = \alpha^\mp(u)$$

Construction of representation in L^2 space:

Consider $L^2(S^1)$: $x(u) = \sum_n x_{-n} e^{inu}$

$$B_K(x,x) = \frac{1}{2} \sum_{n \geq 1} \xi_n^{-1} x_n x_{-n} \qquad \sum_{n=1}^{\infty} \xi_n < \infty$$

The operator K, defined by $\{\xi_n\}$ is trace-class

Gaussian measure:

$$dw_K = \left(\sqrt{\det 2\pi N_K}\right)^{-1} e^{-B_K(x,x)}\, d\phi \prod_{n=1}^{\infty} \left[\frac{i}{2} dx_n \wedge dx_{-n}\right]$$

$$b_{-n} = i(\partial_n - \xi_n^{-1} x_{-n}) \qquad a_{-n} = i\partial_n$$

$$a_n^* = b_{-n} \qquad h_n = \frac{1}{2}(a_n + b_n)$$

Realization of currents:

$$h(u) = \sum_{n=-\infty}^{\infty} h_n e^{-inu} \qquad \alpha^\pm(u) = e^{\pm i x_c(u)} = e^{\pm i\phi + \sum_n x_{-n} e^{inu}}$$

$$h_0 = i\partial_\phi$$

Correlators: $\langle T_1 \dots T_n \rangle = \langle v_0, T_1 \dots T_n v_0 \rangle$

a_n, b_n – annihilation and creation operators

Namely: $\langle T_1 \dots T_n a_k \rangle = 0$, $\langle b_k T_1 \dots T_n \rangle = 0$

$$\langle \alpha_+(u_1) \dots \alpha_+(u_n)\, \alpha_-(v_1) \dots \alpha_-(v_m) \rangle =$$

$$= \delta_{n,m} \exp\Big(-\sum_{i<j} N_k(u_i, u_j) - \sum_{i<j} N_k(v_i, v_j) + \sum_{i,j} N_k(u_i, v_j) + n N_k(0,0)\Big)$$

$$N_k(u,v) = 2 \sum_{n>0} \cos(n(u-v))\, \xi_n$$

Notice: $a_k v_0 = 0$ $\quad \{ b_{m_1} \dots b_{m_s} \alpha^{\pm}_{n_1} \dots \alpha^{\pm}_{n_r} v_0 \}$ span the representation

In addition: $\rho(u) = \sum_n \rho_n e^{-inu}$, $[\rho_n, \rho_m] = 2 k n \delta_{n,-m}$

$$F_{k,p} = \{ \rho_{-n_1} \dots \rho_{-n_k} \mathrm{vac}_p \,;\, n_1 \dots n_k > 0,\ \rho_0 \mathrm{vac}_p = p\, \mathrm{vac}_p \}$$

Regularized currents: $|z|<1$

$$\phi(u) \to \phi(z,\bar z) = \sum_{n\geq 0} \varphi_n \bar z^{n} + \sum_{n>0} \varphi_{-n} z^n \qquad (\varphi = a, b, x^c, \rho)$$

$$J^{\pm}(z,\bar z) = \frac{i}{2}\left(b(z,\bar z)\,\alpha^{\pm}(z,\bar z) + \alpha^{\pm}(z,\bar z)\,a(z,\bar z)\right) \pm k\,\partial_u \alpha^{\pm}(z,\bar z) \pm \rho(z,\bar z)\,\alpha^{\pm}(z,\bar z)$$

$$J^3(z,\bar z) = -2i\,h(z,\bar z) + 2k\,\alpha^{-}(z,\bar z)\,\partial_u \alpha^{+}(z,\bar z)$$

$$J^3(z,\bar z) = -J^3(z,\bar z) \qquad J^{\pm}(z,\bar z) = -J^{\mp}(z,\bar z)$$

Proposition: Correlators

$$\langle \phi_1(z_1,\bar z_1) \;\ldots\; \phi_n(z_n,\bar z_n)\rangle =$$

$$= \langle V_0 \otimes vac_\rho,\ \phi_1(z_1,\bar z_1)\;\ldots\;\phi_n(z_n,\bar z_n)\, V_0 \otimes vac_\rho\rangle$$

are well-defined when $0<|z_i|<1$. Moreover, they are well-defined when one of $|z_i|=1$.

Commutator:

$$\lim_{r_1,r_2\to 1} \langle \ldots \big(\xi(w_1,\bar w_1)\,\eta(w_2,\bar w_2) - \eta(w_2,\bar w_2)\,\xi(w_1,\bar w_1)\big)\ldots\rangle$$

$$w_i = r_i e^{iu_i}$$

Commutation relations:

$$[J^3(u), J^\pm(v)] = \pm 2i\, J^\pm(v)\,\delta(u-v)$$

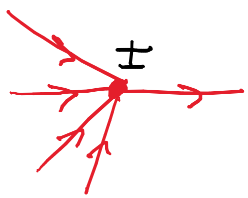
$$[J^+(u), J^-(v)] = i J^3(v)\,\delta(u-v) + 4ik\,\delta'(u-v)$$

Now let us study correlators graphically. Arranging creation and annihilaton operators we obtain commutators $[a(z,\bar z), \alpha^\pm(w,\bar w)] = \mp\,\alpha^\pm(w,\bar w)\,\delta(z,w)$

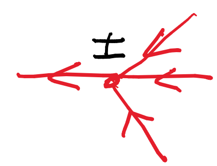
$$[\alpha^\pm(w,\bar w), b(z,\bar z)] = \pm\,\alpha^\pm(w,\bar w)\,\delta(z,w)$$

"propagator" $\xrightarrow{\delta(z,w)}$

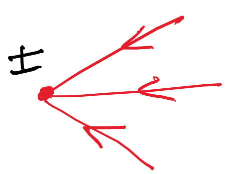
Vertices:



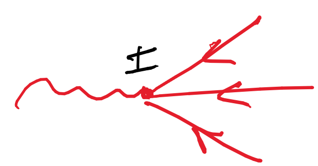
$\alpha^\pm(z,\bar z)\, a(z,\bar z)$



$b(z,\bar z)\,\alpha^\pm(z,\bar z)$

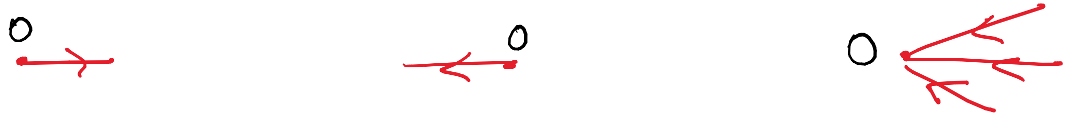


$k\,\partial_u \alpha^\pm(z,\bar z)$



$\rho(z,\bar z)\,\alpha^\pm(z,\bar z)$

"Neutral" vertices:



$\frac{1}{2} a(z,\bar{z})$ $\qquad$ $\frac{1}{2} b(z,\bar{z})$ $\qquad$ $2\kappa\, \alpha^-(z,\bar{z})\partial_u \alpha^+(z,\bar{z})$

Divergence problem: $\langle \dots J^+(z_1,\bar{z}_1)\, J^-(z_2,\bar{z}_2) \dots \rangle$

$\delta(u_1-u_2)$

$\delta(u_1-u_2)$

Loop diagrams: $\delta(z_1,z_2)\delta(z_2,z_3)\dots\delta(z_k,z_1)$

Renormalization: $\mu_k\, \delta(u_1-u_2)\delta(u_2-u_3)\dots\delta(u_{k-1}-u_k)$

Theorem: Regularized correlators

$$\langle \phi_1(z_1,\bar{z}_1) \dots \phi_n(z_n,\bar{z}_n) \rangle_p^{R,\{\mu_n\}}$$

define $\widehat{sl}(2,\mathbb{R})$ module with a Hermitean form, parametrized by the parameter p from the Fock module and regularization parameters $\{\mu_n\}$.

Open questions

i) Which $\{\mu_i\}$ give unitary modules?

ii) Intertwiners / tensor product?

iii) Modular double structure?
(Possibly on the level of intertwiners)

iv) Relations to physical WZW model?