q-Virasoro algebra and affine Lie algebras

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I. Introduction

In vertex algebra theory, among important problems are to develop the theory of vertex operator algebras and their representations, classify irreducible (non-twisted and twisted) modules for certain VOAs, and establish C_2-cofiniteness and rationality. Also, seek for interesting applications in other fields.

To these ends, various new tools, including generalizations of the Zhu algebra $A(V)$ and bimodules $A(W)$, were developed.

On the other hand, a conceptual problem is to establish natural associations of various (Lie or associative) algebras with vertex algebras or their generalizations.

In this direction, new theories, including that of vertex Γ-algebras, quantum vertex algebras, and quasi modules and ϕ-coordinated modules for quantum vertex algebras, have been developed. This talk will be along this line.

An overlook (Summary):

The key idea is to investigate possible algebraic structures "generated" by vertex operators=field operators =current operators on a vector space W.

Two main factors for vertex operators:

I) Shape: Formal integer power series in one variable

$$a(x) \in \mathrm{Hom}(W, W((x))) =: \mathcal{E}(W).$$

II) Compatibility (among vertex operators):

i) Locality: $a(x)$ and $b(x)$ are local if there exists $k \geq 0$ such that

$$(x-z)^k a(x)b(z) = (x-z)^k b(z)a(x).$$

This leads to vertex algebras and modules.

ii) Quasi locality: $a(x)$ and $b(x)$ are quasi local if there exists a nonzero polynomial $p(x,z)$ such that

$$p(x,z)a(x)b(z) = p(x,z)b(z)a(x).$$

This leads to vertex algebras and quasi modules.

iii) S-locality: A subset U of $\mathcal{E}(W)$ is S-local if for any $a(x), b(x) \in U$, there exist

$$c_i(x),\ d_i(x) \in U, \quad f_i(x) \in \mathbb{C}((x)),\ i = 1, \ldots, r$$

and a nonnegative integer k such that

$$(x-z)^k a(x)b(z) = (x-z)^k \sum_{i=1}^{r} f_i(z-x)c_i(z)d_i(x).$$

This leads to quantum vertex algebras and modules.

iv) Quasi S-locality, which leads to quantum vertex algebras and quasi modules.

v) S_{trig}-locality: A subset U of $\mathcal{E}(W)$ is S_{trig}-local if for any $a(x), b(x) \in U$, there exist

$$c_i(x),\ d_i(x) \in U, \quad g_i(x) \in \mathbb{C}(x),\ i = 1, \ldots, r$$

and a nonnegative integer k such that

$$(x - z)^k a(x)b(z) = (x - z)^k \sum_{i=1}^{r} g_i(x/z)c_i(z)d_i(x).$$

This leads to quantum vertex algebras and ϕ-coordinated modules.

vi) Quasi S_{trig}-locality, which leads to quantum vertex algebras and quasi ϕ-coordinated modules.

Note: Local twisted vertex operators, which are fraction power series, lead to vertex algebras and twisted modules.

This talk is mainly on quasi locality and quasi modules for vertex algebras, where we use q-Virasoro algebra as a concrete example to illustrate this theory.

II. Modules for vertex algebras and locality

Let V be a vertex algebra. A V-module is a vector space W with a linear map

$$Y_W(\cdot, x) : V \to (\mathrm{End}\, W)[[x, x^{-1}]]$$

such that $Y_W(\mathbf{1}, x) = 1_W$, and for $v \in V$,

$$Y_W(v, x)w \in W((x)) \quad \text{for } w \in W,$$

and Jacobi identity holds for $u, v \in V$:

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u, x_1) Y_W(v, x_2) \\ & \quad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v, x_2) Y_W(u, x_1) \\ = \; & x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(u, x_0)v, x_2). \end{aligned}$$

Jacobi identity is equivalent to Locality: For $u, v \in V$, there exists $k \geq 0$ such that

$$(x-z)^k Y_W(u,x) Y_W(v,z) = (x-z)^k Y_W(v,z) Y_W(u,x)$$

and Weak associativity: For $u, v, w \in V$, there exists $l \geq 0$ such that

$$(x+z)^l Y_W(u, x+z) Y_W(v,z) w = (x+z)^l Y_W(Y(u,z)v, x) w.$$

In fact, for vertex algebra modules, weak associativity alone is sufficient.

Let W be a vector space. Set

$$\mathcal{E}(W) = \mathrm{Hom}(W, W((x))).$$

Locality: A subset S of $\mathcal{E}(W)$ is said to be local if for any $A(x), B(x) \in S$, there exists a nonnegative integer k such that

$$(x_1 - x_2)^k A(x_1)B(x_2) = (x_1 - x_2)^k B(x_2)A(x_1).$$

Operations on $\mathcal{E}(W)$: For $A(x), B(x) \in \mathcal{E}(W)$ and for $n \in \mathbb{Z}$, we define $A(x)_n B(x) \in \mathcal{E}(W)$ by

$$\begin{aligned} & (A(x)_n B(x))w \\ = \ & \mathrm{Res}_{x_1}(x_1 - x)^n A(x_1)B(x)w - (-x + x_1)^n B(x)A(x_1)w \end{aligned}$$

for $w \in W$.

Theorem (L)

Let S be a local subset of $\mathcal{E}(W)$. Denote by $\langle S \rangle$ the linear span of

$$A^{(1)}(x)_{n_1} \cdots A^{(r)}(x)_{n_r} 1_W$$

for $A^{(i)}(x) \in S$, $n_i \in \mathbb{Z}$. Then $\langle S \rangle$ is a vertex algebra and W is an $\langle S \rangle$-module.

III. Quasi modules for vertex algebras and quasi locality

Let W be a vector space. Recall $\mathcal{E}(W) = \mathrm{Hom}(W, W((x)))$

A subset U of $\mathcal{E}(W)$ is quasi-local if for any $a(x), b(x) \in U$, there exists a nonzero polynomial $p(x, y)$ such that

$$p(x_1, x_2)a(x_1)b(x_2) = p(x_1, x_2)b(x_2)a(x_1).$$

Note that this equality implies

$$p(x_1, x_2)a(x_1)b(x_2) \in \mathrm{Hom}(W, W((x_1, x_2))).$$

Adjoint vertex operator map $Y_{\mathcal{E}}$ on $\mathcal{E}(W)$

Let $a(x), b(x) \in \mathcal{E}(W)$. Assume that there exists a nonzero polynomial $p(x, z)$ such that

$$p(x_1, x_2)a(x_1)b(x_2) \in \mathrm{Hom}(W, W((x_1, x_2))).$$

Define $a(x)_n b(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ in terms of

$$Y_{\mathcal{E}}(a(x), z)b(x) = \sum_{n \in \mathbb{Z}} a(x)_n b(x) z^{-n-1}$$

by

$$Y_{\mathcal{E}}(a(x), z)b(x) = p(x+z, x)^{-1} \left(p(x_1, x)a(x_1)b(x)\right)|_{x_1 = x+z},$$

where $p(x+z, x)^{-1}$ denotes the inverse in $\mathbb{C}((x))((z))$.

Note: The essence of this definition is OPE.

Theorem (L)

Every quasi local subset of $\mathcal{E}(W)$ generates a vertex algebra with W a faithful quasi module in the following sense.

Let V be a vertex algebra. A quasi V-module is defined simply by replacing the Jacobi identity in the definition of a module with a weaker identity: For any $u, v \in V$, there exists a nonzero polynomial $p(x, z)$ such that the Jacobi identity after multiplied by $p(x_1, x_2)$ holds.

Note: Often, to better describe the vertex algebras generated by quasi local subsets one needs to consider $a(\lambda x)$ with $\lambda \in \mathbb{C}^\times$. Consequently, the vertex algebras we obtain naturally come with a group action.

Vertex Γ-algebras and equivariant quasi modules

Let Γ be a group equipped with a linear character χ. A vertex Γ-algebra is a vertex algebra V on which Γ acts such that $R_g(\mathbf{1}) = \mathbf{1}$,

$$R_g Y(v,x) R_g^{-1} = Y(R_g v, \chi(g)^{-1} x) \quad \text{for } g \in \Gamma.$$

Let V be a vertex Γ-algebra. A (Γ,χ)-equivariant quasi V-module is a quasi V-module (W, Y_W), satisfying

$$Y_W(R_g v, x) = Y_W(v, \chi(g)x) \quad \text{for } g \in \Gamma, \ v \in V$$

and for any $u, v \in V$, there exists a polynomial

$$p(z) \in \langle z - \chi(g) \mid g \in \Gamma \rangle$$

such that the Jacobi identity multiplied by $p(x_1/x_2)$ holds.

Two ways to get vertex Γ-algebras

Let Γ be a subgroup of $\mathbb{C}^\times$. A subset U of $\mathcal{E}(W)$ is Γ-local if for any $a(x), b(x) \in U$, there is a polynomial $p(z)$ of the form $(z-\alpha_1)\cdots(z-\alpha_r)$ with $\alpha_i \in \Gamma$ such that

$$p(x/z)a(x)b(z) = p(x/z)b(z)a(x).$$

Set

$$U_\Gamma = \{a(\lambda x) \mid a(x) \in U,\ \lambda \in \Gamma\},$$

which is also Γ-local.

Theorem (L)

For every Γ-local subset U of $\mathcal{E}(W)$, the vertex algebra $\langle U_\Gamma \rangle$ generated by U_Γ is a vertex Γ-algebra with W a faithful equivariant quasi module.

A $\mathbb{Z}$-graded vertex algebra is a vertex algebra V equipped with a $\mathbb{Z}$-grading $V = \oplus_{n \in \mathbb{Z}} V_{(n)}$ such that

$$u_k V_{(n)} \subset V_{(m+n-k-1)}$$

for $u \in V_{(m)},\ m, n, k \in \mathbb{Z}$.

Let Γ be an automorphism group of V, preserving the $\mathbb{Z}$-grading, and let χ be any linear character. For $g \in \Gamma$, set

$$R_g = \chi(g)^{-L(0)} g,$$

where $L(0)$ is the linear operator on V, defined by $L(0)|_{V_{(n)}} = n$ for $n \in \mathbb{Z}$. Then V becomes a vertex Γ-algebra.

Equivariant quasi modules and twisted modules

Let V be a VOA with an automorphism σ of order T. Set $G = \langle \sigma \rangle$ and let χ be the linear character given by $\chi(\sigma) = e^{2\pi i/T}$. Then V becomes a vertex G-algebra.

Theorem (L)

The category of σ-twisted V-modules is canonically isomorphic to the category of equivariant quasi V-modules.

The essence of the proof is change-of-coordinate: $z \to z^T$. A result of Barron-Dong-Mason was used in an essential way.

Note: In view of this, the notion of equivariant quasi module generalizes that of twisted module.

Let K be a Lie algebra and let G be an automorphism group. Then one has an (orbifold) Lie subalgebra

$$K^G = \{a \in K \mid g(a) = a \quad \text{for } g \in G\}.$$

Fact: Twisted affine Kac-Moody algebras can be realized as orbifold subalgebras of untwisted affine Kac-Moody algebras with respect to dynkin diagram automorphisms.

The following is another way to associate a Lie algebra to a pair (K, G) as above (cf. [L]).

Lemma

Let K be a Lie algebra with a symmetric invariant bilinear form $\langle\cdot,\cdot\rangle$ and let G be an automorphism group of K, preserving $\langle\cdot,\cdot\rangle$, such that for any $a, b \in K$,

$$[ga, b] = 0 \quad \text{and} \quad \langle ga, b\rangle = 0 \qquad \textit{for all but finitely many } g \in G.$$

Define a new operation $[\cdot,\cdot]_G$ and a new form $\langle\cdot,\cdot\rangle_G$ on K by

$$[a, b]_G = \sum_{g\in G}[ga, b] \quad \text{and} \quad \langle a, b\rangle_G = \sum_{g\in G}\langle ga, b\rangle.$$

Set

$$I_G = \mathrm{span}\{a - ga \mid a \in K,\ g \in G\}.$$

Then I_G is an ideal of the non-associative algebra $(K, [\cdot,\cdot]_G)$ and the quotient algebra K/I_G is a Lie algebra. Furthermore, $\langle\cdot,\cdot\rangle_G$ reduces to a symmetric invariant bilinear form on K/I_G.

The Lie algebra obtained above is alternatively denoted by K/G and called the G-covariant algebra of K.

Fact: If G is finite, then K/G is isomorphic to K^G.

Let $\mathfrak{g}$ be a (possibly infinite-dimensional) Lie algebra with a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ and let G an automorphism group of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ such that for any $a, b \in \mathfrak{g}$,

$$[\sigma a, b] = 0 \text{ and } \langle \sigma a, b \rangle = 0 \quad \text{for all but finitely many } \sigma \in G.$$

Use χ to lift G to an automorphism group of $\hat{\mathfrak{g}}$. Let $V_{\hat{\mathfrak{g}}}(\ell, 0)$ be the universal affine vertex algebra associated to $\hat{\mathfrak{g}}$ of level ℓ.

Theorem (L)

Let χ be a faithful linear character of G. There is a canonical isomorphism from the category of restricted $\hat{\mathfrak{g}}/G$-modules of level ℓ to that of (G, χ)-equivariant quasi $V_{\hat{\mathfrak{g}}}(\ell, 0)$-modules.

IV. The q-Virasoro algebra

Based on a joint work with H. Guo, S. Tan, Qing Wang.

The q-Virasoro algebra D, introduced by Belov and Chaltikian, is the Lie algebra with generators $\mathbf{c}$ and $D^{\alpha}(n)$ with $\alpha, n \in \mathbb{Z}$, subject to relations $D^{-\alpha}(n) = -D^{\alpha}(n)$ and

$$\begin{aligned}[D^{\alpha}(n), D^{\beta}(m)] &= (q - q^{-1})[\alpha m - \beta n]_q D^{\alpha+\beta}(m+n) \\ &\quad - (q - q^{-1})[\alpha m + \beta n]_q D^{\alpha-\beta}(m+n) \\ &\quad + ([m]_{q^{\alpha+\beta}} - [m]_{q^{\alpha-\beta}})\delta_{m+n,0}\mathbf{c}\end{aligned}$$

for $\alpha, \beta, m, n \in \mathbb{Z}$, where $\mathbf{c}$ is a central element, q is a nonzero complex parameter.

Here, we define a Lie algebra D_S associated to an abelian group S with a faithful linear character χ by replacing q^{α} with $\chi(\alpha)$ in the structural coefficients.

For each $\alpha \in S$, we first form a generating function

$$D^{\alpha}(x) = \sum_{n \in \mathbb{Z}} D^{\alpha}(n) x^{-n-1}.$$

However, the commutator formula for $D^{\alpha}(x)$ with $\alpha \in S$ is not closed to a certain sense. For this reason, we modify the generating functions as follows:

$$\widetilde{D}^{\alpha}(x) = \begin{cases} D^{\alpha}(x) & \text{if } 2\alpha = 0 \\ D^{\alpha}(x) - \frac{1}{\chi(-\alpha) - \chi(\alpha)} \mathbf{c} x^{-1} & \text{if } 2\alpha \neq 0. \end{cases}$$

The defining relations of D_S are equivalent to

$$\tilde{D}^{-\alpha}(x) = -\tilde{D}^{\alpha}(x),$$

$$\begin{aligned}
& [\tilde{D}^{\alpha}(x_1), \tilde{D}^{\beta}(x_2)] \\
= \; & \chi(-\alpha)\tilde{D}^{\alpha+\beta}(\chi(-\alpha)x_2)x_1^{-1}\delta\left(\frac{\chi(-\alpha-\beta)x_2}{x_1}\right) \\
& -\chi(\alpha)\tilde{D}^{\alpha+\beta}(\chi(\alpha)x_2)x_1^{-1}\delta\left(\frac{\chi(\alpha+\beta)x_2}{x_1}\right) \\
& -\chi(-\alpha)\tilde{D}^{\alpha-\beta}(\chi(-\alpha)x_2)x_1^{-1}\delta\left(\frac{\chi(\beta-\alpha)x_2}{x_1}\right) \\
& +\chi(\alpha)\tilde{D}^{\alpha-\beta}(\chi(\alpha)x_2)x_1^{-1}\delta\left(\frac{\chi(\alpha-\beta)x_2}{x_1}\right) \\
& -\chi(\alpha-\beta)\delta_{2(\alpha-\beta),0}\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{\chi(\alpha-\beta)x_2}{x_1}\right)\mathbf{c} \\
& +\chi(\alpha+\beta)\delta_{2(\alpha+\beta),0}\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{\chi(\alpha+\beta)x_2}{x_1}\right)\mathbf{c}.
\end{aligned}$$

For $\alpha, \beta \in \mathbb{Z}$, we have

$$p(x_1, x_2)[\tilde{D}^{\alpha}(x_1), \tilde{D}^{\beta}(x_2)] = 0$$

with

$$p(x_1, x_2) = (x_1 - q^{\alpha+\beta}x_2)(x_1 - q^{-\alpha-\beta}x_2)(x_1 - q^{\alpha-\beta}x_2)(x_1 - q^{\beta-\alpha}x_2).$$

They are quasi local, though not local.

These generating functions on a restricted D_S-module W generate a vertex Γ-algebra with W as an equivariant quasi module.

This indicates that D_S should be a "twisted Lie algebra." Next, we shall use the general machinery to obtain its "untwisted partner," by examining the vertex Γ-algebra.

Define a new Lie algebra $\mathfrak{g}_S$ generated by $d^{\alpha,r}$ for $\alpha \in S$, $r \in \mathbb{Z}$, subject to relations: $d^{-\alpha,r} = -d^{\alpha,r}$ and

$$\begin{aligned}[d^{\alpha,r}, d^{\beta,s}] &= \delta_{\alpha+\beta,s-r} d^{\alpha+\beta,-\alpha+s} - \delta_{\alpha+\beta,r-s} d^{\alpha+\beta,\alpha+s} \\ &\quad - \delta_{\alpha-\beta,s-r} d^{\alpha-\beta,-\alpha+s} + \delta_{\alpha-\beta,r-s} d^{\alpha-\beta,\alpha+s}.\end{aligned}$$

On $\mathfrak{g}_S$, there is a non-degenerate symmetric invariant bilinear form $\langle \cdot,\cdot \rangle_\chi$ defined by

$$\begin{aligned}&\langle d^{\alpha,\beta}, d^{\mu,\nu} \rangle_\chi \\ = \;& \chi(\alpha+\mu)\delta_{2(\alpha+\mu),0}\delta_{\alpha+\mu,\beta-\nu} - \chi(\alpha-\mu)\delta_{2(\alpha-\mu),0}\delta_{\alpha-\mu,\beta-\nu}\end{aligned}$$

for $\alpha, \beta, \mu, \nu \in S$.

Then we have an affine Lie algebra $\widehat{\mathfrak{g}_S} = \mathfrak{g}_S \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c}$. Furthermore, for any complex number ℓ, we have a $\mathbb{Z}$-graded vertex algebra $V_{\widehat{\mathfrak{g}_S}}(\ell, 0)$, whose underlying vector space is the level ℓ generalized Verma module (or Weyl module) of $\widehat{\mathfrak{g}_S}$.

We can lift S to be an automorphism group of affine Lie algebra $\hat{\mathfrak{g}}_S$ and the $\mathbb{Z}$-graded vertex algebra $V_{\widehat{\mathfrak{g}_S}}(\ell, 0)$. Then $V_{\widehat{\mathfrak{g}_S}}(\ell, 0)$ becomes a vertex S-algebra.

We have the following results ([Guo-L-Tan-Wang]):

Theorem

The linear map $\pi : \widehat{\mathfrak{g}_S} \to D_S$, *defined by*

$$\pi(d^{\alpha,\beta}(x)) = D^{\alpha,\beta}(x) = \chi(\beta)\tilde{D}^{\alpha}(\chi(\beta)x) \quad \textit{for } \alpha, \beta \in S,$$

gives rise to a Lie algebra isomorphism from the covariant algebra $\widehat{\mathfrak{g}_S}/S$ *of the affine Lie algebra* $\widehat{\mathfrak{g}_S}$ *to* D_S.

Theorem

The category of restricted D_S*-modules of level* ℓ *is naturally isomorphic to the category of equivariant quasi* $V_{\widehat{\mathfrak{g}_S}}(\ell, 0)$*-modules.*

Theorem

Assume that S *is a finite abelian group of order* $2l+1$. *Then* D_S *is isomorphic to the untwisted affine Kac-Moody algebra of type* $B_l^{(1)}$.

Thank You