The constraint manifold of general relativity

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UC, Irvine and Stanford

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Marsden Memorial Lecture

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Plan of Lecture

The lecture will have four parts:

Part 1: Review of the Einstein equations

Part 2: Linearization stability and its generalizations

Part 3: Overview of results

Part 4: Sketches of proofs and further questions

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Marsden co-authors in this field include: J. Arms, D. Bao, M. Cantor, D. Eardley, A. Fischer, M. Gotay, J. Isenberg, V. Moncrief, R. Montgomery, N. Ó Murchadha, F. Tipler, R. Walton, P. Yasskin, and J. York

Part 1: Review of the Einstein equations

On a spacetime $\mathcal{S}^{n+1}$, the Einstein equations couple the gravitational field g (a Lorentz metric on $\mathcal{S}$) with the matter fields via their stress-energy tensor T

$$Ric(g) - \frac{1}{2} R\, g = T$$

where Ric denotes the Ricci curvature and $R = Tr_g(Ric(g))$ is the scalar curvature.

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where Ric denotes the Ricci curvature and $R = Tr_g(Ric(g))$ is the scalar curvature.

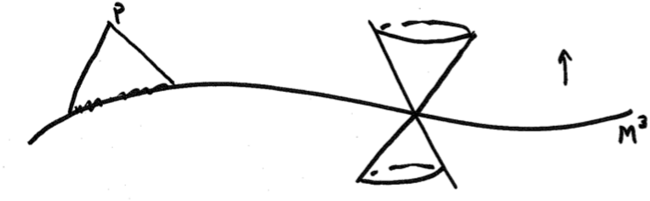
When there are no matter fields present the right hand side T is zero, and the equation reduces to

$$Ric(g) = 0.$$

These equations are called the vacuum Einstein equation.

Initial Data

The solution is determined by initial data given on a spacelike hypersurface M^n in $\mathcal{S}$.



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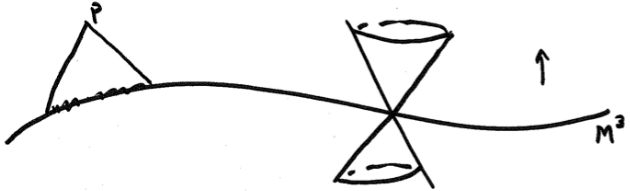

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The initial data for g are the induced (Riemannian) metric, also denoted g, and the second fundamental form p. These play the role of the initial position and velocity for the gravitational field. An initial data set is a triple (M, g, p).

The constraint equations

It turns out that $n+1$ of the $(n+1)(n+2)/2$ Einstein equations can be expressed entirely in terms of the initial data and so are not dynamical. These come from the Gauss and Codazzi equations of differential geometry.

In case there is no matter present, the vacuum constraint equations become

$$R_M + Tr_g(p)^2 - \|p\|^2 = 0$$
$$\sum_{j=1}^{n} \nabla^j \pi_{ij} = 0$$

for $i = 1, 2, \ldots, n$ where R_M is the scalar curvature of M and $\pi_{ij} = p_{ij} - Tr_g(p) g_{ij}$.

The initial value problem

Given an initial data set (M, g, p) satisfying the vacuum constraint equations, there is a unique maximal globally hyperbolic spacetime which evolves from that data. This result involves the local solvability of a system of nonlinear wave equations.

Boundary conditions: Compact Cauchy surface

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The compactness often makes the analysis easier, so this is a positive feature. On the other hand it is harder to interpret quantities such as gravitational energy and momentum in this setting.

Asymptotically flat manifolds

An important case for us is the asymptotically flat case. The requirement is that the initial manifold M outside a compact set be diffeomorphic to the exterior of a ball in R^n and that there be coordinates x in which g and p have appropriate falloff.

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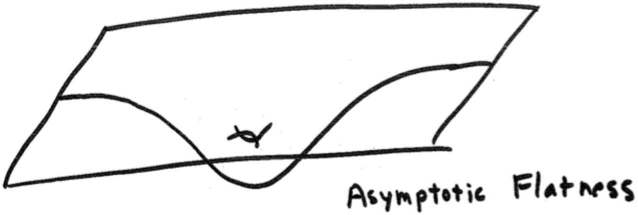

Minkowski and Schwarzschild Solutions

The following are two basic examples of asymptotically flat spacetimes:

1) The Minkowski spacetime is R^{n+1} with the flat metric $g = -dx_0^2 + \sum_{i=1}^{n} dx_i^2$. It is the spacetime of special relativity.

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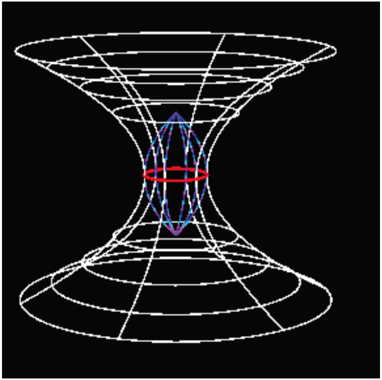
2) The Schwarzschild spacetime is determined by initial data with $p = 0$ and

$$g_{ij} = (1 + \frac{E}{2|x|^{n-2}})^{\frac{4}{n-2}} \delta_{ij}$$

for $|x| > 0$. It is a vacuum solution describing a static black hole with mass E. It is the analogue of the exterior field in Newtonian gravity induced by a point mass.

The Schwarzschild spacetime

Here is a picture of the extended Schwarzschild initial manifold. Its features lead to important notions for general asymptotically flat solutions such as the ADM energy-momentum and the notions of black holes and trapped surfaces.



Asymptotically hyperbolic manifolds

If one thinks of asymptotic flatness as the condition that the spacetime approaches the Minkowski space at spatial infinity, there is an analogous notion for spacetimes which approach the Minkowski spacetime at null infinity. These are spacetimes generated by initial data which is asymptotic to the hyperbolic space at infinity.

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1) The first example is the hyperboloid in Minkowski spacetime

$$x^{n+1} = \sqrt{1 + |x'|^2} \quad \text{where} \quad x' = (x^1, \ldots, x^n).$$

The metric is the hyperbolic metric, so $R = -n(n-1)$ and it is umbilic with $p = g$.

The asymptotically hyperbolic Schwarzschild spacetime

2) There is also an asymptotically hyperbolic slice of the Schwarzschild solution which has metric given in spherical coordinates (r, ξ)

$$g = (1 + r^2 - 2Er^{-1})^{-1} dr^2 + r^2 d\xi^2, \text{ with } r \geq 2E$$

and with $p = g$.

Part 2: Linearization stability and its generalizations

Notice that the initial value problem allows us to parametrize solutions of the Einstein equations by solutions of the constraint equations. On the other hand we may think of the constraint 'manifold' as the set $\Phi(g, p) = 0$ where (g, p) consist of a metric and a symmetric $(0, 2)$ tensor on a given manifold M. Notice that the domain of Φ is an open set of a vector space. The map Φ is the constraint map

$$\Phi(g, p) = (R(g) + Tr_g(p)^2 - \|p\|^2, \sum_{j=1}^{n} \nabla^j \pi_{ij})$$

where $\pi = p - Tr_g(p)g$.

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where $\pi = p - Tr_g(p)g$.

A solution of the Einstein equations is said to be *linearization stable* if every infinitesimal deformation is tangent to a family of deformations.

A simple example

Let $\Phi(x, y) = x^2 - y^2$ defined on $\mathbb{R}^2$ and consider the set $\Sigma = \{\Phi = 0\}$. At a point $(x, y) \in \Sigma$ the space of infinitesimal deformations consists of the kernel of $d\Phi$ at the point (x, y). For points $(x, y) \neq (0, 0)$ this defines the tangent line to Σ and each such vector is tangent to a curve in Σ.

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At $(0, 0)$ we have $d\Phi \equiv 0$, and every vector at this point is an infinitesimal deformation. The only vectors which are tangent to curves in Σ are those which make a 45^0 angle with the coordinate axes.

Linearization stability

As in the example, one expects the constraint manifold to be smooth at a solution (g, p) if (g, p) is linearization stable. In fact Marsden and his collaborators showed that linearization stability is the necessary and sufficient condition for smoothness of the constraint manifold near a given point. In order to formulate this it is necessary to be precise about topologies on the space of tensors (g, p).

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The question of integrating infinitesimal deformations comes up in any problem involving a moduli space of solutions. Linearization stability is the condition that this can be done for all infinitesimal deformations.

Linearization stability and symmetry

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Marsden and collaborators showed that for compact manifolds linearization stability fails at (g, p) if and only if the spacetime generated has a Killing vector field.

They also gave a condition which characterizes those infinitesimal deformations which are integrable. It is the vanishing of a certain second order conserved quantity found by A. Taub. Thus they showed that the singularities of the constraint manifold are quadratic in nature (like the simple example).

Extensions of the idea; localization of supports

A natural question to ask is whether it is possible to integrate infinitesimal deformations at a point (g, p) which are supported in some open set Ω of M in such a way that the resulting path of solutions is equal to (g, p) outside Ω.

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A natural question to ask is whether it is possible to integrate infinitesimal deformations at a point (g, p) which are supported in some open set Ω of M in such a way that the resulting path of solutions is equal to (g, p) outside Ω.

It turns out that if (g, p) is locally linearization stable in the sense that there are no spacetime Killing vector fields in Ω, there are methods to achieve this. We will describe methods for doing this sort of construction.

Enlarging the class of deformations

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For example if we have a submanifold Σ given by $\Phi = 0$ in $\mathbb{R}^n$ and a point $p \in \Sigma$, we could determine a path in Σ by taking a path $p + tv$ in $\mathbb{R}^n$, and projecting it to Σ using a local retraction from a neighborhood of Σ to Σ.

Enlarging the class of deformations

Combining these ideas, we can consider taking a point (g, p) in the constraint manifold and perturbing it in the space of tensors to a nearby $(\tilde{g}, \tilde{p})$ which may not be in the constraint manifold. We can then attempt to project into the constraint manifold to obtain $(\hat{g}, \hat{p})$ which is near (g, p) and satisfies the constraints.

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Assuming that the initial perturbation $(\tilde{g}, \tilde{p})$ agrees with (g, p) outside some open set Ω we can ask that $(\hat{g}, \hat{p})$ also remains unperturbed outside Ω.

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A natural application of this idea would be to simplify the geometry of (g, p) inside an open set Ω without changing it outside Ω. We will see some applications of this in the next part of the talk.

Understanding the obstructions

The fact that it is not generally possible to do this is illustrated by the following example. Let (g, p) be the euclidean metric on $\mathbb{R}^3$ and $p = 0$, so that it is initial data for Minkowski space. Now take symmetric $(0, 2)$ tensors h and k to have compact support. Let $\tilde{g} = g + \epsilon h$ and $\tilde{p} = \epsilon k$ for ϵ small.

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It is not possible to perturb to $(\hat{g}, \hat{p})$ satisfying the constraint equations and having compact support because this would lead to a compactly supported solution of the constraint equations violating the positive energy theorem.

It turns out that one can account for the obstruction by allowing flexibility in the exterior solution one uses, so that by allowing an exterior Schwarzschild or Kerr solution the construction can be made.

Part 3: Overview of results: simplifying solutions

As far as we know, the first instance of this general technique was used by the speaker and Yau in the early 1980s to simplify the asymptotics of general asymptotically flat metrics with $p = 0$. The idea is, given g a general asymptotically flat metric with $R = 0$, we let

$$\tilde{g} = \chi g + (1 - \chi)\delta$$

where χ is a cutoff function which is 1 in a large ball and 0 outside a ball of twice the radius.

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where χ is a cutoff function which is 1 in a large ball and 0 outside a ball of twice the radius.

We then use conformal deformation to construct a metric $\hat{g} = u^4 \tilde{g}$ to impose the constraint equations $\hat{R} = 0$. In this way we can approximate g by a solution which is conformally flat outside a compact set. This can be done for virtually any metric g for which the ADM energy exists.

The spacetime case

There is a spacetime ($p \neq 0$) analogue of this result. Notice that the metric $u^4\delta$ has zero scalar curvature if and only if u is harmonic. Since harmonic functions have nice asymptotic behavior, it is natural to look for solutions of the general constraint equations which are given by harmonic functions near infinity.

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If we require the conditions

$$g = u^4\delta, \text{ and } \pi = u^2[L_X g - div_g(X)g],$$

then to leading order the constraint equations imply that u and the components of the vector field X are harmonic.

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It was shown by Corvino and the speaker that solutions with this asymptotic form are dense in the constraint manifold of asymptotically flat solutions. With this asymptotic behavior the ADM conserved quantities appear as terms in the asymptotic expansion.

Specifying precise asymptotic behavior

Since it is possible to achieve any chosen pair E, P by a suitably boosted slice in the Schwarzschild, people have assumed that this would be a natural asymptotic form for an asymptotically flat solution of the vacuum constraint equations.

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It was shown by J. Corvino ($p = 0$) and by Corvino and S. (also Chruściel and Delay) that the set of initial data which are identical to a boosted slice of the Kerr (generalization of Schwarzschild) spacetime are dense in a natural topology in the space of all data with reasonable decay.

Localization of initial data

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A good linear analogue is data (E, B) for the Maxwell equations on $\mathbb{R}^3$ with $B = 0$ and E satisfying the constraint equation $divE = 4\pi q$ where q is a compactly supported charge density. There is then a total charge and so the solution cannot be approximated by solutions of compact support. There is considerable flexibility in the asymptotic form of E which can be achieved.

Asymptotic behavior

The energy and linear momentum can be shown to exist under very weak asymptotic decay

$$g_{ij} = \delta_{ij} + O_2(|x|^{-q}), \; p_{ij} = O_1(|x|^{-q-1})$$

for any $q > (n-2)/2$.

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In order to understand the global properties of the Einstein evolution it is important to understand what asymptotic form is reasonable to assume. The positive energy theorem implies that there are no solutions of the constraint equations with compact support.

A further consequence of positive energy

If we let U denote the open subset of M consisting of those points at which the Ricci curvature of g is nonzero, then we have the following. It shows that under reasonable decay conditions the set U must include a positive 'angle' at infinity.

Proposition Assume that (M, g, p) satisfies the decay conditions

$$g_{ij} = \delta_{ij} + O_2(|x|^{2-n}),\ p_{ij} = O_1(|x|^{1-n}).$$

Unless the initial data is trivial, we have

$$\liminf_{\sigma \to \infty} \sigma^{1-n} Vol(U \cap \partial B_\sigma) > 0.$$

Proof of proposition

The energy can be written in terms of the Ricci curvature

$$E = -c_n \lim_{\sigma\to\infty} \sigma \int_{S_\sigma} Ric(\nu,\nu)\ da$$

for a positive constant c_n. If our initial data is nontrivial, then we have $E > 0$, and so for any σ sufficiently large we have

$$E/2 < c_n\sigma \int_{S_\sigma} |Ric(\nu,\nu)|\ da \leq c\sigma^{1-n} Vol(U \cap \partial B_\sigma)$$

where the second inequality follows from the decay assumption.

□

Localizing solutions in a cone

Let us consider an asymptotically flat manifold (M, g) with $R_g = 0$ and with decay

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In joint work with A. Carlotto we have shown that there is a metric $\hat{g}$ which satisfies $R_{\hat{g}} = 0$ with $\hat{g} = g$ inside a cone based at a point far out in the asymptotic region while $\hat{g} = \delta$ outside a cone with slightly larger angle. Moreover $\hat{g}$ is close to g in a topology in which the energy is continuous, so $\hat{E}$ is arbitrarily close to E. The metric $\hat{g}$ satisfies

$$\hat{g}_{ij} = \delta_{ij} + O(|x|^{-q})$$

provided $q < n-2$.

arXiv:1407.4766

Where is the energy?

Since there is very little contribution to the energy inside the region where $\bar{g} = g$ and none in the euclidean region, most of the energy resides on the transition region. This shows that one cannot impose too much decay on this region and makes the weakened decay plausible.

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There is a recent localization theorem in the asymptotically hyperbolic case by P. Chrusciel and E. Delay (arXiv:1511.07858) in which they show that solutions can be localized in certain regions extending to conformal infinity. In the upper half space model such regions include a half ball centered at a point on the boundary.

Part 4: Sketches of the proof and further questions

For simplicity we will just consider the case of the constraints when $p = 0$. We consider a region Ω and suppose we have a metric $\tilde{g}$ which has zero scalar curvature outside Ω.

We then seek a solution of the form $\hat{g} = \tilde{g} + h$ where h vanishes in a neighborhood of $\partial\Omega$ (and outside Ω) with $R(\hat{g}) = 0$. The equation can be written

$$R(\hat{g}) = R(\tilde{g}) + \tilde{L}h + Q(h) = 0$$

where $\tilde{L}$ is the linearization of the scalar curvature map at $\tilde{g}$.

The equations

We have the formula for the operator

$$\tilde{L}h = \delta\delta h - \Delta_{\tilde{g}}(Tr(h)) - \langle h, Ric(\tilde{g})\rangle$$

where computations are with respect to $\tilde{g}$. The adjoint operator is then

$$\tilde{L}^* u = Hess_{\tilde{g}}(u) - \Delta_{\tilde{g}}(u)\tilde{g} - uRic(\tilde{g}).$$

The composition is given by

$$\begin{aligned}\tilde{L}(\tilde{L}^* u) = (n-1)\Delta(\Delta u) + 1/2(\Delta \tilde{R})u + 3/2\langle \nabla \tilde{R}, \nabla u\rangle \\ + 2\tilde{R}(\Delta u) - \langle Hess(u), Ric(\tilde{g})\rangle\end{aligned}$$

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We thus need to be able to solve the problem

$$\tilde{L}h + Q(h) = f$$

in appropriate spaces.

The compact unobstructed case

In this case we solve the equation

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using a Picard iteration scheme in spaces which impose decay of ϕ^N near $\partial\Omega$ where N is chosen large and ϕ is a function which behaves like distance to the boundary. The proof involves first showing that $\tilde{L}$ is surjective in such spaces.

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The basic estimate which enables us to impose rapid decay near $\partial\Omega$ is

$$\|u\|_{2,\Omega} \leq c\|\tilde{L}^* u\|_{0,\Omega}$$

where these are norms in L^2 Sobolev spaces and no boundary condition is imposed on u.

The compact obstructed case

This includes the case of approximating solutions by exact Kerr solutions near infinity. In this case the basic estimate is the same but only holds for u which are orthogonal to the kernel of the adjoint operator. This space is finite dimensional and can be explicitly identified. By considering a family of approximate solutions with varying values of the ADM conserved quantities (energy, linear and angular momentum, and center of mass) it is possible to arrange that the component of the equation along the cokernel also vanishes.

The noncompact case

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In all of the cases we must also handle the spacetime case ($p \neq 0$). This complicates the problem in all cases, but it is especially so in the noncompact case since the equations we consider are of mixed order in addition to the domain being noncompact.

Avoiding derivative loss

A natural way to solve the linear equation $\tilde{L}h = f$ is to look for a solution of the form $\tilde{L}^* u$ and to observe that the operator $\tilde{L}\tilde{L}^*$ is a self adjoint fourth order operator with leading term the bi-harmonic operator. This idea goes back to a 1975 paper of A. Fischer and J. Marsden. It has a drawback in that the lower order terms of the operator $\tilde{L}\tilde{L}^*$ involve four derivatives of $\tilde{g}$, and so the solution $h = \tilde{L}^*(u)$ is two derivatives less smooth than $\tilde{g}$.

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This derivative loss problem was pointed out to the speaker 25 years ago by R. Bartnik. Fischer and Marsden get around this problem by considering only C^∞ metrics and working in a Frechet space. It is desirable to prove the results with a fixed finite degree of differentiability, and this can be done in the global compact setting considered by Fischer and Marsden by working with respect a fixed smooth background metric. We expect that this can also be done in the more complicated cases involving localized deformations.

Thank you for listening!