Asymptotic behaviour of the Hawking energy in null directions and a Penrose-like inequality

Marc Mars (joint work with Alberto Soria)

Universidad de Salamanca

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Penrose inequality for initial data sets

- Consider asymptotically flat initial data sets $(\mathcal{N}^3, \gamma, K)$ (one end) satisfying DEC.
- Σ weakly outer trapped: $\theta_+ := H_\Sigma + \mathrm{tr}_\Sigma K \leq 0$.
- Future trapped region $\mathcal{T}_\Sigma^+ \subset \Sigma$: Union of compact domains with weakly outer trapped boundary.

Conjecture (Penrose inequality)

Let $(\mathcal{N}^n, g, K)$ be an asymptotically flat initial data set satisfying DEC and $S_{\min}(\partial \mathcal{T}_\Sigma^+)$ the minimal area enclosure of $\partial \mathcal{T}_\Sigma^+$. Then

$$(2M_{ADM})^{\frac{n-1}{n-2}} \geq \frac{S_{\min}(\partial \mathcal{T}_\Sigma^+)}{\omega_{n-1}}, \qquad \omega_{n-1} = |\mathbb{S}^{n-1}|$$

Moreover, equality implies $(\Sigma \setminus \mathcal{T}_\Sigma^+, g, K)$ can be isometrically embedded into the Schwarzschild spacetime.

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Moreover, equality implies $(\Sigma \setminus \mathcal{T}_\Sigma^+, g, K)$ can be isometrically embedded into the Schwarzschild spacetime.

Proven in the time symmetric case $K = 0$:

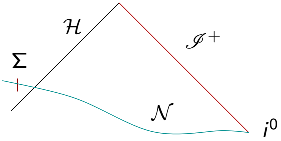
- In full generality for $3 \leq n \leq 7$ ([Huisken & Ilmanen '01], [Bray '01], [Bray & Lee '09]).
- For graphical manifolds in $\mathbb{E}^{n+1}$, $n \geq 3$ ([Lam '10], [Huang & Wu '12]).

Still not much known when $K \neq 0$.

Penrose inequality for null hypersurfaces

Need for $S_{\min}(\partial\mathcal{T}_\Sigma^+)$ comes from the heuristics behind the Penrose inequality (restrict to $n = 4$)

- The standard collapse scenario implies
$$16\pi M_{ADM}^2 \geq |\mathcal{H} \cap \mathcal{N}|.$$
- $\mathcal{H}$ event horizon of the black hole that forms (weak cosmic censorship).
- $\partial\mathcal{T}_\Sigma^+$ is known to lie inside the black hole (but may have larger area than $\mathcal{H} \cap \mathcal{N}$.



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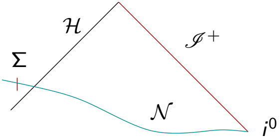
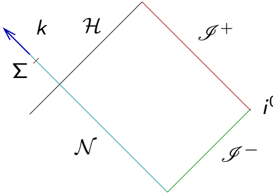


The minimal area enclosure takes care of this: $16\pi M_{ADM}^2 \geq |\mathcal{H} \cap \mathcal{N}| \geq S_{\min}(\partial\mathcal{T}_\Sigma^+)$

There are situations where $S_{\min}$ is not necessary.

- Assume (M, g) admits a past null infinity $\mathscr{I}^-$.
- Consider a smooth null hypersurface $\mathcal{N}$ extending to $\mathscr{I}^-$ and containing a weakly trapped surface.
- Smoothness of $\mathcal{N}$ requires $\theta_k \leq 0 \quad \Longrightarrow \quad |\mathcal{H} \cap \mathcal{N}| \geq |\Sigma|$



Penrose heuristics gives:

$$16\pi M_B(\mathcal{N})^2 \geq |\Sigma|, \qquad M_B(\mathcal{N}) \text{ Bondi mass of } \mathcal{N}.$$

Leads to a Penrose inequality conjecture for null hypersurfaces.

- Can be formulated in any spacetime dimension.

Conjecture (Null Penrose inequality)

Let $\mathcal{N}$ be an asymptotically flat null hypersurface in a spacetime satisfying DEC. Assume that $\Sigma \hookrightarrow \mathcal{N}$ is a weakly outer trapped surface. Then, the Bondi mass of $\mathcal{N}$ satisfies

$$M_B(\mathcal{N}) \geq \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

Inequality involves:
- Intrinsic and extrinsic geometry of $\mathcal{N}$.
- Asymptotic conditions along $\mathcal{N}$.

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Inequality involves:
- Intrinsic and extrinsic geometry of $\mathcal{N}$.
- Asymptotic conditions along $\mathcal{N}$.

- No need for minimal area enclosures.

Need to define:

- Asymptotic flatness.
- Bondi mass (or Bondi energy).

Penrose's original version of the inequality involved a particular case of this (shells propagating in Minkowski).

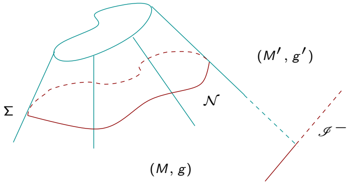
Consequences of the Null Penrose inequality conjecture

Null Penrose inequality assumes a weakly outer trapped surface.

- However, it has implications on general asymptotically flat vacuum spacetimes.

The physical idea is to let shells (distributional matter) propagate on a given spacetime.

- This can be made precise. Assume (M, g):
  - vacuum
  - admitting a null AF hypersurface $\mathcal{N}$



Select any cross section Σ on $\mathcal{N}$

- Modify appropriately the characteristic data on $\mathcal{N}$, so that it stays vacuum.
- Suplement with data on $\mathscr{I}^-$

Another asymptotically flat vacuum spacetime (M', g') exists.

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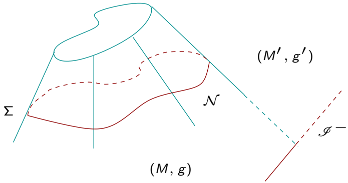

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- Modify appropriately the characteristic data on $\mathcal{N}$, so that it stays vacuum.
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Another asymptotically flat vacuum spacetime (M', g') exists.

- Data can always be arranged so that Σ is a weakly outer trapped surface in (M', g').
- Bondi energy of (M', g') can be computed in terms of the Bondi energy of (M, g) and the geometry of $\Sigma \hookrightarrow (M, g)$

The Penrose inequality applied to (M', g') leads to a geometric inequality purely (M, g)

Theorem (Shell-Penrose inequality, [M., 2016])

- *Let $\mathcal{N}$ be an asymptotically flat null hypersurface embedded in a vacuum spacetime (M, g) and Σ any cross section of $\mathcal{N}$.*
- *Let $\{\Sigma_\lambda\}$ by a foliation by cross sections starting at Σ, approaching large spheres and with a geodesic flow vector k.*
- *Let ℓ be the future null normal to Σ satisfying $\langle k, \ell \rangle = -2$.*

If the null Penrose inequality conjecture holds, then the Bondi energy E_B associated to $\{\Sigma_\lambda\}$ satisfies

$$E_B + \frac{1}{16\pi} \int_\Sigma \theta_\ell \boldsymbol{\eta}_\Sigma \geq \sqrt{\frac{|\Sigma|}{16\pi}}, \tag{1}$$

where θ_ℓ is the null expansion of Σ along ℓ in (M, g).

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Can be extended to other matter models.

- Proving (1) for specific spacetimes (e.g. Minkowski) is still a hard problem.

Known results on the null Penrose inequality

The null Penrose inequality has been proven in a number of special cases:

- General proof was claimed by [Ludvigsen & Vickers, '83].
    - Important gap found by [Bergvist, '97].
- For shear-free null hypersurfaces ($K^k \propto$ first fund. form γ) [Sauter, Ph.D. thesis 2008]
    - The null Penrose inequality reduces to

$$\int_{\mathbb{S}^2} (F^2 + |dF|^2_{\mathring{q}})\, \boldsymbol{\eta}_{\mathring{\mathbf{q}}} \geq \sqrt{4\pi \int_{\mathbb{S}^2} F^4\, \boldsymbol{\eta}_{\mathring{\mathbf{q}}}}.$$

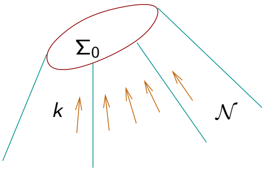
    - Particular case of a general Sobolev-type inequality on $\mathbb{S}^n$ [Beckner '93]
- For null shells propagating in Minkowski in special cases [Tod, '85], [Gibbons, '97], [M. & Soria, 14], [M. & Soria, 15].
- For shells propagating in Schwarzschild [Brendle & Wang, 14].
- For vacuum spacetimes near Schwarzschild [Alexakis '15].

Asymptotically flat null hypersurface

We want a definition that involves local "in time" conditions on $\mathcal{N}$ (global along $\mathcal{N}$).

Convenient to use foliations of $\mathcal{N}$ by spacelike surfaces.

- Let $k \in \Gamma(T\mathcal{N})$ be future null and nowhere zero (null generator)
- Satisfies $\nabla_k k = Q_k k, \quad Q_k : \mathcal{N} \mapsto \mathbb{R}$
- k is called geodesic if $Q_k = 0$.



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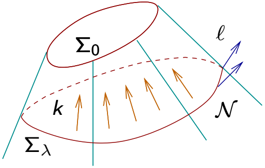

Definition ($\mathcal{N}$ extending to past null infinity)

A null hypersurface $\mathcal{N}$ in a spacetime (M, g) is extends to past null infinity if

- $\mathcal{N}$ admits a cross section Σ_0 of spherical topology.
- Affinely parametrized null geodesics starting at $p \in \Sigma_0$ with tangent vector $-k|_p$ have maximal domain $(\lambda_0(p), \infty)$.

- Chose k geodesic. Define $\lambda : \mathcal{N} \mapsto \mathbb{R}$ by $k(\lambda) = -1$ and $\lambda|_{\Sigma_0} = 0$.
- Level sets Σ_λ of λ are cross sections of spherical topology.
  - In particular $\mathcal{N} = \mathbb{R} \times \mathbb{S}^2$

Define ℓ along $\mathcal{N}$ by: (i) ℓ null, (ii) orthogonal to Σ_λ, (iii) $\langle k, \ell \rangle = -2$

We use λ to specify the decay at infinity.

A covariant tensor field T on $\mathcal{N}$ is

- Transversal: if $T(\cdots, k, \cdots) = 0$.
- Lie constant: if $\mathcal{L}_k T = 0$.

Transversal tensors are in one-to-one correspondence to a family $T(\lambda)$ on Σ_λ.

- Denote $T_{A_1 \cdots A_q} := T(X_{A_1}, \cdots X_{A_q})$
  X_A local basis of $T\Sigma_0$ extended to $\mathcal{N}$ by $[k, X_A] = 0$.

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Definition

A transversal tensor field T on $\mathcal{N}$ is

- $T = O(1)$ if $T_{A_1 \cdots A_q}$ is uniformly bounded.
- $T = O_n(\lambda^{-q})$, $(q, n \in \mathbb{N})$ iff $\lambda^{q+i} (\mathcal{L}_k)^i T = O(1)$, $i = 0, \cdots, n$.
- $T = o(\lambda^{-q})$ and $T = o_n(\lambda^{-q})$ defined similarly.

Given a transversal T, $\mathcal{L}_{X_A} T$ is also transversal.

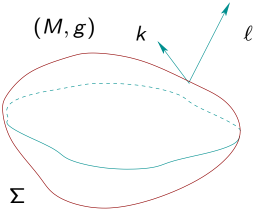
- $T = o^X_n(\lambda^{-q})$ iff

$$\lambda^q \underbrace{\mathcal{L}_{X_{A_1}} \cdots \mathcal{L}_{X_{A_i}}}_{i} T = o(1) \qquad \forall i = 0, 1, \cdots, n \qquad \text{for all values of } A_1, \cdots, A_i.$$

Definitions independent of the choice of λ and the choice of basis $\{X_A\}$

Notation for spacelike codimension-two surfaces:

- Induced metric: γ
- Mean curvature $\vec{H}$ (outwards for a sphere)
- Null expansions $\theta_k := \langle k, \vec{H} \rangle$, $\theta_\ell := \langle \ell, \vec{H} \rangle$,
- Normal connection $s_\ell(X) = -\frac{1}{2} \langle \nabla_X \ell, k \rangle$.



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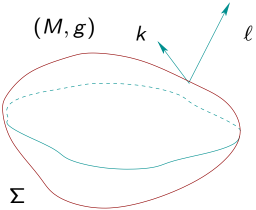

Definition (Asymptotically flat $\mathcal{N}$)

A null hypersurface Ω in a spacetime (M, g) is past asymptotically flat if (i) extends to past null infinity and (ii) for a choice of geodesic k and corresponding level function λ:

(i) The first fundamental form γ of $\mathcal{N}$ is $\gamma = \lambda^2 \hat{q} + \lambda h + \tilde{\gamma}$:
$\hat{q}, h$ transversal and Lie-constant, $\hat{q} > 0$, $\tilde{\gamma} = o_1(\lambda) \cap o_2^X(\lambda)$.

(ii) The normal connection s_ℓ of $\{\Sigma_\lambda\}$ is $s_\ell = s_\ell^{(1)} \lambda^{-1} + o_1(\lambda^{-1})$, $s_\ell^{(1)}$ Lie constant

(iii) The null expansion θ_ℓ is $\theta_\ell = \theta_\ell^{(0)} \lambda^{-1} + \theta_\ell^{(1)} \lambda^{-2} + o(\lambda^{-2})$: $\theta_\ell^{(0)}, \theta_\ell^{(1)}$ Lie constant,

(iv) $\lim_{\lambda \to \infty} \lambda^{-2} \mathrm{Riem}^g(X_A, X_B, X_C, X_D)$ exists and its double trace satisfies

$$2\mathrm{Ein}^g(k, \ell) - \mathrm{Scal}^g - \frac{1}{2}\mathrm{Riem}^g(\ell, k, \ell, k) = o(\lambda^{-2}).$$

Consequences: $\theta_\ell^{(0)} = \frac{2\mathcal{K}_{\hat{q}}}{\lambda} + \frac{\theta_\ell^{(1)}}{\lambda^2} + o(\lambda^{-2})$, $\theta_k = -\frac{2}{\lambda} + \frac{\theta_k^{(1)}}{\lambda^2} + o(\lambda^{-2})$

Bondi energy and Hawking energy

$\hat{q}$ can be thought of as the metric of Σ_λ "at infinity".

- Under a rescaling $k' = f\,k$, with $f > 0$ Lie constant: $\hat{q}' = f^2 \hat{q}$.
- Always exists a choice of geodesic k such that $\hat{q}$ is the standard metric on $\mathbb{S}^2$.
    - Denoted by $\mathring{q}$.

Foliation Σ_λ associated to such k: approaching large spheres Not unique.

Any such choice of geodesic foliation defines an observer at infinity.

The Bondi energy is a quantity associated to a past asymptotically flat null hypersurface $\mathcal{N}$ for any choice of observer at infinity.

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The Bondi energy is a quantity associated to a past asymptotically flat null hypersurface $\mathcal{N}$ for any choice of observer at infinity.

- Recall the Hawking energy of a spacelike surface

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma \langle \vec{H}, \vec{H} \rangle \eta_\Sigma \right).$$

Definition (Bondi energy)

Let $\mathcal{N}$ be a past asymptotically flat null hypersurface. Select a geodesic null generator k and let the associated foliation $\{\Sigma_\lambda\}$ approach large spheres.
The Bondi energy of the observed at infinity defined by this foliation is

$$E_B(\mathcal{N}) := \lim_{\lambda \to \infty} m_H(\Sigma_\lambda).$$

Limit of the Hawking energy at infinity along general foliations

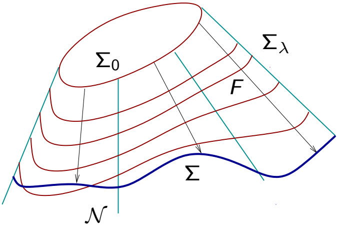
- Geodesic foliations approaching large spheres are very special.

Aim: understand the asymptotic value of the Hawking energy along general foliations.

Let $\mathcal{N}$ be null past asymptotically flat, k geodesic and Σ_λ an associated foliation.

- Any cross section can be defined as the graph $\{\lambda = F\}$, of a function $F : \Sigma_0 \longrightarrow \mathbb{R}$
- Foliations by cross sections can be defined in terms of one-parameter families of function $F_{\lambda^\star} : \Sigma_0 \to \mathbb{R}$

$$\Sigma_{\lambda^\star} := \text{graph}(F_{\lambda^\star})$$



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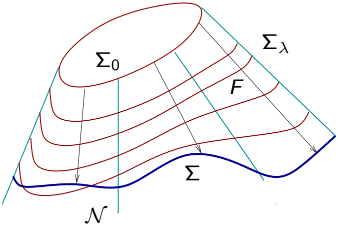

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Examples:

- $F_{\lambda^\star} = \phi\lambda^\star$ with $\phi : \Sigma_0 \to \mathbb{R}^+$:
  Geodesic foliation with the same initial surface Σ_0 and different speed ($k^\star = \phi k$).
- $F_{\lambda^\star} = \lambda^\star + \tau$ with $\tau : \Sigma_0 \to \mathbb{R}$:
  Geodesic foliation with initial surface $\Sigma_0^\star = \text{graph}(\tau)$ and same speed.
- $F_{\lambda^\star} = \lambda^\star + \xi(\lambda^\star)$ with ξ suitably decaying:
  Non-geodesic foliation which approaches $\{\Sigma_\lambda\}$ at infinity.

A combination describes any foliation with reaches infinity non-zero speed (vector flow bounded away from zero).

Asymptotics of the Hawking energy

Theorem (M. & Alberto Soria, '2015)

- *Let Ω be a past asymptotically flat null hypersurface in spacetime (M, g).*
- *Select a geodesic generator k with corresponding foliation $\{\Sigma_\lambda\}$ approaching large spheres.*
- *Consider another foliation $\{\Sigma_{\lambda^\star}\}$ defined by the the level sets of the function $\mathcal{F}(\mathcal{N}) \ni \lambda^\star := \Psi\lambda - \tau - \xi$ where*

$$\tau, \Psi > 0 \in \mathcal{F}(\mathcal{N}), \text{ Lie constant}, \qquad \xi = o_1(1) \cap o_2^X(1) \text{ and } k(\xi) = o_1^X(\lambda^{-1})$$

The limit of the Hawking energy along $\{\Sigma_{\lambda^\star}\}$ is

$$\lim_{\lambda^\star \to \infty} m_H(\Sigma_{\lambda^\star}) = \frac{1}{8\pi} \left(\sqrt{\int_{\mathbb{S}^2} \frac{1}{16\pi\Psi^2} \boldsymbol{\eta}_{\mathring{\mathbf{q}}}} \right) \int_{\mathbb{S}^2} \left(\triangle_{\mathring{q}} \theta_k^{(1)} - (\theta_k^{(1)} + \theta_\ell^{(1)}) - 4 \mathrm{div}_{\mathring{q}}(s_\ell^{(1)}) \right) \Psi \boldsymbol{\eta}_{\mathring{\mathbf{q}}},$$

where $\theta_k^{(1)}$, $\theta_\ell^{(1)}$, $s_\ell^{(1)}$ and $\mathring{q}$ refer to the background foliation $\{\Sigma_\lambda\}$.

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where $\theta_k^{(1)}$, $\theta_\ell^{(1)}$, $s_\ell^{(1)}$ and $\mathring{q}$ refer to the background foliation $\{\Sigma_\lambda\}$.

- Remarkably simple dependence on Ψ, τ and ξ.
- Integrand with interesting invariance properties under change of geodesic foliation.
- Recovers the limit to the Bondi energy when $\{\Sigma_\lambda\}$ approaches large spheres.

An approach to the null Penrose inequality

- Key object: Functional on spacelike surfaces. $M(\Sigma, \ell) = \sqrt{\frac{|\Sigma|}{16\pi}} - \frac{1}{16\pi} \int_\Sigma \theta_\ell \boldsymbol{\eta_\Sigma}$,
  - Physical dimensions of length (energy), but not truly quasi-local (there is ℓ).
- However, on a weakly outer trapped surface ($\theta_\ell \leq 0$) satisfies $M(\Sigma, \ell) \geq \sqrt{\frac{|\Sigma|}{16\pi}}$.
- It may interpolate between both sides of the null Penrose inequality
  - Need to understand its monotonicity properties and its asymptotic behaviour.

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Lemma

- *Let $\mathcal{N}\simeq\mathbb{R}\times\Sigma$ be a null hypersurface with generator along $\mathbb{R}$.*
- *Let Σ_λ be a foliation by cross sections and scale the null generator so that $k(\lambda)=-1$. Define Q_k by $\nabla_k k=Q_k k$.*
- *For any choice of function $\varphi>0$ on Σ_λ let ℓ^φ be null and normal with $\langle k,\ell^\varphi\rangle=-\varphi$.*

Then:

$$\frac{dM(\Sigma_\lambda,\ell^\varphi)}{d\lambda}=\frac{1}{\sqrt{64\pi|\Sigma_\lambda|}}\int_{\Sigma_\lambda}(-\theta_k)\boldsymbol{\eta_{\Sigma_\lambda}}+\frac{1}{16\pi}\int_{\Sigma_\lambda}\Big[\mathrm{Ein}^g(\ell,k)-\frac{\varphi}{2}\mathrm{Scal}^{\Sigma_\mu}$$
$$+\varphi\left(-div_{\Sigma_\lambda}s_{\ell^\varphi}+|s_{\ell^\varphi}|^2_{\gamma_{\Sigma_\lambda}}\right)+\left(\frac{1}{\varphi}k(\varphi)-Q_k\right)\theta_{\ell^\varphi}\Big]\boldsymbol{\eta_{\Sigma_\lambda}}$$

Leads naturally to $\varphi=$ const and $Q_k=0$.

For $\varphi =$ and $Q_k = 0$:

$$\frac{dM(\Sigma_\lambda, \ell^\varphi)}{d\lambda} = \frac{\int_{\Sigma_\lambda}(-\theta_k)\boldsymbol{\eta}_{\boldsymbol{\Sigma}_{\boldsymbol{\lambda}}}}{\sqrt{64\pi|\Sigma_\lambda|}} + \frac{1}{16\pi}\int_{\Sigma_\lambda}\left(\mathrm{Ein}^g(\ell^\varphi, k) + \varphi|s_{\ell^\varphi}|^2_{\gamma_{\Sigma_\lambda}}\right)\boldsymbol{\eta}_{\boldsymbol{\Sigma}_{\boldsymbol{\lambda}}} - \frac{\varphi\,\chi(\Sigma_\lambda)}{8}.$$

- Monotonic under DEC if the (connected) Σ has non-zero genus.
- Non-monotonic in the spherical case.
    - However, there is only one bad term.

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- Monotonic under DEC if the (connected) Σ has non-zero genus.
- Non-monotonic in the spherical case.
    - However, there is only one bad term.
- Concerning the asymptotic behaviour.

Lemma

- *Let $\mathcal{N}$ be past asymptotically flat, $\{\Sigma_\lambda\}$ a geodesic foliation and $\varphi > 0$ a constant.*
- *The limit $\lim_{\lambda \to \infty} M(\Sigma_\lambda, \ell^\varphi)$ is finite if and only if $\varphi = 2R_{\hat{q}}$ with $R_{\hat{q}} := \sqrt{\frac{|\Sigma|_{\hat{q}}}{4\pi}}$.*

Let $\ell^\star = R_{\hat{q}} \ell$. Then $\lim_{\lambda \to \infty} M(\Sigma_\lambda, \ell^\star) = \lim_{\lambda \to \infty} m_H(\Sigma_\lambda) + \frac{1}{16\pi} \int_\Sigma \theta_k^{(1)} \left(\frac{1}{R_{\hat{q}}} - R_{\hat{q}} \mathcal{K}_{\hat{q}} \right) \boldsymbol{\eta}_{\hat{\mathbf{q}}}.$

Two interesting cases where the two limits agree:

- $\hat{q}$ has constant curvature, since then $\mathcal{K}_{\hat{q}} = 1/R^2_{\hat{q}}$ (approach to large spheres).
- When $\theta_k^{(1)}$ is constant. By Gauss-Bonnet $\int_\Sigma \mathcal{K}_{\hat{q}} \boldsymbol{\eta}_{\hat{\mathbf{q}}} = 4\pi$.

Geodesic asymptotically Bondi foliations

Definition

Let $\mathcal{N}$ be a past asymptotically flat null hypersurface and Σ_0 be a cross section.
A geodesic foliation $\{\Sigma_\lambda\}$ is called geodesic, asymptotically Bondi and associated to Σ_0 iff

(i) $\Sigma_{\lambda=0} = \Sigma_0$.

(ii) With k the associated null generator ($k(\lambda) = -1$), the leading term $\theta_k^{(1)}$ in

$$\theta_k = -\frac{2}{\lambda} + \frac{\theta_k^{(1)}}{\lambda^2} + o(\lambda^{-2}) \qquad \text{is constant.}$$

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Proposition (Existence and uniqueness)

Assume DEC. Given any cross section Σ_0:

(i) *There exists a geodesic asymptotically Bondi foliation associated to Σ_0.*

(ii) *This foliation is unique except for trivial constant reparametrizations $\lambda' = a\lambda$, $a \in \mathbb{R}$*

- The functional $M(\Sigma, \ell^{\varphi})$ is not monotonic in the spherical case.

To approach the null Penrose inequality (or variations) we need less:

- Bound from above $M(\Sigma_0, \ell^{\star})$ at the initial weakly outer trapped surface

$$M(\Sigma_0, \ell^{\star}) \leq \lim_{\lambda \to \infty} M(\Sigma_\lambda, \ell^{\star})$$

To exploit the good terms in the variation formula:

- Split $M(\Sigma, \ell^{\varphi})$ in two: $$M(\Sigma, \ell^{\varphi}) := \underbrace{\left(\sqrt{\frac{|\Sigma|}{16\pi}} - \frac{\varphi}{4}\lambda \right)}_{:= D(\Sigma, \ell^{\varphi})} + \underbrace{\left(\frac{\varphi}{4}\lambda - \frac{1}{16\pi} \int_{\Sigma} \theta_{\ell^{\varphi}} \boldsymbol{\eta}_{\boldsymbol{\Sigma}} \right)}_{M_b(\Sigma, \ell^{\varphi})}$$
- $M_b(\Sigma, \ell^{\varphi})$ (introduced by [Bergqvist, '97]): Monotonic for geodesic null flows + DEC.
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Monotonicity of M_b implies automatically $M_b(\Sigma_0, \ell^{\star}) \leq \lim_{\lambda \to \infty} M(\Sigma_\lambda, \ell^{\star})$

- Need to understand under which conditions

$$D(\Sigma_0, \ell^{\star}) \leq \lim_{\lambda \to \infty} D(\Sigma_\lambda, \ell^{\star}).$$

Any situation where such bound holds leads immediately to a Penrose-like inequality

A Penrose-like inequality

- Geodesic asymptotically Bondi (GAB) foliations lead to a Penrose-like inequality

Proposition

Let $\mathcal{N}$ be null and past asymptotically flat. Let $\{\Sigma_\lambda\}$ be a GAB foliation. Then

$$D(\Sigma_0, \ell^\star) \leq \lim_{\lambda \to \infty} D(\Sigma_\lambda, \ell^\star)$$

- Combining with the limit at infinity of $M(\Sigma_\lambda, \ell^\star)$.

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- Combining with the limit at infinity of $M(\Sigma_\lambda, \ell^\star)$.

Theorem ([M. & A. Soria, '16])

- *Let $\mathcal{N}$ be a past asymptotically flat null hypersurface and Σ_0 a cross section.*
- *Assume DEC.*

Then

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} - \frac{1}{16\pi} \int_{\Sigma_0} \theta_{\ell^\star} \boldsymbol{\eta}_{\boldsymbol{\Sigma_0}} \leq \lim_{\lambda \to \infty} m_H(\Sigma_\lambda),$$

where the limit is taken along the GAB foliation $\{\Sigma_\lambda\}$ associated to Σ_0.

In particular, if Σ_0 is weakly outer trapped: $\sqrt{\frac{|\Sigma_0|}{16\pi}} \leq \lim_{\lambda \to \infty} m_H(\Sigma_\lambda)$.

This is the null Penrose inequality whenever, in addition, the GAB foliation Σ_λ approaches large spheres.

Key ingredient in the proof:

$$F(\Sigma_\lambda) = \frac{|\Sigma_\lambda|}{\left(8\pi R_{\hat{q}}^2 \lambda + \int_\Sigma \theta_k^{(1)} \boldsymbol{\eta}_{\hat{\mathbf{q}}}\right)^2}$$ monotonically increasing for GAB foliations + DEC.

- Monotonicity of $F(\Sigma_\lambda)$ is useful for general geodesic foliations because (with no additional assumptions):

$$\frac{dF(\Sigma_\lambda)}{d\lambda} \geq 0 \quad \Longrightarrow \quad D(\Sigma_0, \ell^\star) \leq \lim_{\lambda\to\infty} D(\Sigma_\lambda, \ell^\star).$$

Can one find conditions ensuring monotonicity of $D(\Sigma_\lambda, \ell)$ or $F(\Sigma_\lambda)$ in the case of foliations approaching large spheres? Renormalized area method

Key ingredient in the proof:

$$F(\Sigma_\lambda) = \frac{|\Sigma_\lambda|}{\left(8\pi R^2_{\hat{q}}\lambda + \int_\Sigma \theta^{(1)}_k \boldsymbol{\eta}_{\hat{\mathbf{q}}}\right)^2}$$ monotonically increasing for GAB foliations + DEC.

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Can one find conditions ensuring monotonicity of $D(\Sigma_\lambda, \ell)$ or $F(\Sigma_\lambda)$ in the case of foliations approaching large spheres? Renormalized area method

- Need to strengthen slightly the definition of asymptotic flatness.

Definition

A null hypersurface is strongly past asymptotically flat if, in addition to being past asymptotically flat, the first fundamental form of $\mathcal{N}$ admits an expansion

$$\gamma = \lambda^2 \hat{q} + \lambda h + \Psi_0 + o_1(1) \cap o_2^X(1), \qquad \hat{q} > 0, h, \Psi_0 \quad \text{Lie constant.}$$

Consequence: $\theta_k = -\frac{2}{\lambda} + \frac{\theta^{(1)}_k}{\lambda^2} + \frac{\theta^{(2)}_k}{\lambda^2} + o(\lambda^{-2})$.

Sufficient conditions for the renormalized area method

- Studying the variation of $D(\Sigma_\lambda, \ell)$ along the foliation

Theorem ([M. & A. Soria, '16])

Let $\mathcal{N}$ be strong past asymptotically flat null hypersurface and assume DEC. Let $\{\Sigma_\lambda\}$ be a geodesic foliation approaching large spheres.
Assume the two conditions

(i) $$\left(\int_{\mathbb{S}^2} \theta_k^{(1)} \boldsymbol{\eta}_{\mathring{\mathbf{q}}}\right)^2 - 8\pi \int_{\mathbb{S}^2} \left(\theta_k^{(1)}\right)^2 \boldsymbol{\eta}_{\mathring{\mathbf{q}}} - 8\pi \int_{\mathbb{S}^2} \theta_k^{(2)} \boldsymbol{\eta}_{\mathring{\mathbf{q}}} \geq 0$$

(ii) $$\int_{\Sigma_\lambda} \left(-2\theta_k \mathrm{Ric}^g(k,k) + 2(\Pi^k)^{AB} R_{AB} + \frac{d}{d\lambda}\mathrm{Ric}^g(k,k)\right) \boldsymbol{\eta}_{\boldsymbol{\Sigma}_\lambda} \leq 0, \qquad \forall \lambda \geq 0$$

hold, where Π^k is the trace-free part of the null second fundamental form K^k and $R_{AB} = \mathrm{Riem}^g(X_A, k, X_B, k)$. Then

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} - \frac{1}{16\pi}\int_{\Sigma_0} \theta_\ell \, \boldsymbol{\eta}_{\boldsymbol{\Sigma}_0} \leq E_B \qquad E_B \text{ Bondi energy associated to } \{\Sigma_\lambda\}. \tag{2}$$

If, in addition, Σ_0 is weakly outer trapped, the Penrose inequality $E_B \geq \sqrt{\frac{|\Sigma_0|}{16\pi}}$ holds.

Applications

- Method particularly well-adapted to vacuum + shear-free case $\Pi^k = 0$.
    - We can recover and strengthen the result by Sauter.

Theorem ([M. & A. Soria, '16])

- *Let $\mathcal{N}$ be a shear-free, past asymptotically flat null hypersurface in a vacuum (M, g).*
- *Let Σ_0 be a cross section and select k along Σ_0 so that the corresponding geodesic foliation $\{\Sigma_\lambda\}$ approaches large spheres.*
- *Define $F > 0$ by $F^2 = -2(\theta_k|_{\Sigma_0})^{-1}$ and decompose $s_\ell = s_\ell^\perp + \gamma dF$.*

Then, the Bondi energy associated to $\{\Sigma_\lambda\}$ satisfies

$$E_B = \sqrt{\frac{|\Sigma_0|}{16\pi}} - \frac{1}{16\pi}\int_{\Sigma_0} \theta_\ell \boldsymbol{\eta}_{\boldsymbol{\Sigma_0}}$$

$$+\frac{1}{8\pi}\Bigg(\underbrace{\int_{\mathbb{S}^2}\left(F^2 + |dF|^2_{\mathring{q}}\right)\boldsymbol{\eta}_{\mathring{\mathfrak{q}}} - \sqrt{4\pi\int_{\mathbb{S}^2} F^2 \boldsymbol{\eta}_{\mathring{\mathfrak{q}}}}}_{\geq 0 \text{ by Beckner}} + \underbrace{\frac{1}{3}\int_{\mathbb{S}^2} F^2|s_\ell^\perp|^2 + \frac{(1+\gamma F^2)^2}{F^2}|dF|^2 \boldsymbol{\eta}_{\mathring{\mathfrak{q}}}}_{\geq 0} \Bigg).$$

- Method also well-suited for the Minkowski spacetime $\longrightarrow$ Shell-Penrose inequality in Minkowski for a large class of cases.