Improving the Recent Results for the Vacuum Einstein Conformal Constraint Equations by Using the Half-Continuity Method

BIRS Workshop: Geometric Analysis and General Relativity

20 July 2016

Table of Contents :

Part 1.

The Einstein conformal constraint equations

The conformal method

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$$\frac{4(n-1)}{n-2}\Delta\varphi + \mathrm{Scal}\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1} + |\sigma + LW|^2\varphi^{-N-1} \quad (1a)$$

$$-\frac{1}{2}L^*LW = \frac{n-1}{n}\varphi^N d\tau. \quad (1b)$$

Here $N = 2n/(n-2)$ and L is the conformal Killing operator defined by

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Problem : The study of solutions to the conformal constraint equations (1)

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The complete achievements :

- ([Ise95]) The CMC case, i.e. when τ is constant,
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- Existence of solutions to the conformal constraint equations with freely specified mean curvature τ (*the far-from-CMC* case)
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We will summarize recent results on the next slide.

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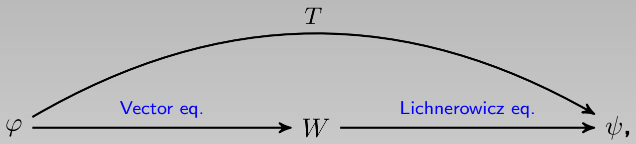

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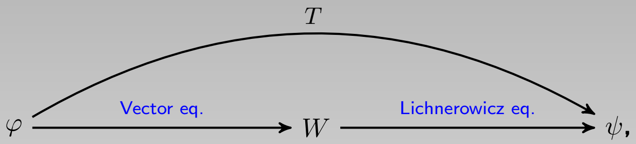

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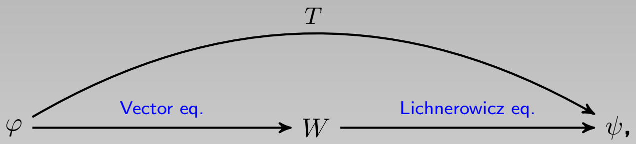

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Proposition 2.1

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Part 2.

The half-continuity method

Idea

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Recall that

Theorem 3.1 (Schaefer's fixed point theorem)

$T : [0,1] \times X \longrightarrow X$ is a continuous compact mapping. At least one of the following assertions is true

(i) $\exists x^* \in X \; : \; x^* = T(1, x^*)$,

(ii) *There exists (t_n, x_n) such that $x_n = t_n T(t_n, x_n)$ and $||x_n|| \to \infty$.*

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Can we replace the assertion (ii) by the following one :

(ii)' *There exists (t_n, x_n) such that $x_n = t_n T(t_n, x_n)$ and x_n satisfies a certain expected property (CEP(x_n) for short).*

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We will provide the so-called *half-continuity method* for addressing the question above.

Half-Continuity

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Definition 3.3

A map $T : C \longrightarrow X$ is said to be half-continuous if for each $x \in C$ with $x \neq T(x)$ there exists $p \in X^*$ and a neighborhood W of x in C such that

$$\langle p, T(y) - y \rangle > 0$$

for all $y \in W$ with $y \neq T(y)$.

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Theorem 3.5 ([TK10] or [Bic06])

Let C be a nonempty closed convex subset of a Banach space X. If $T : C \to C$ is half-continuous and $T(C)$ is precompact, then T has a fixed point.

Technique

Suppose that

$$\text{CEP}(x) \qquad \underleftarrow{\text{is implied by}} \qquad F(t,x) = 0,$$

where the function $F \;:\; [0,1] \times X \longrightarrow \mathbb{R}$ satisfies the following conditions :

(a) F is continuous,

(b) $F(0,0) < 0$,

(c) $\sup \left\{ ||T(t,x)||_{L^\infty} \;:\; F(t,x) \leq 0 \right\} \leq C$.

We will address the question as follows.

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For a given (M, g) with the positive Yamabe invariant

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Question 1 : Is the smallness assumption on σ improved with L^2-norm ?

Answer : YES, it is. We obtain that for a given (M, g) with the positive Yamabe invariant

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Improvement 2 :

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Theorem 3.7 (Dahl–Gicquaud–Humbert [DGH12])

$$\begin{cases} |\tau| > 0 \\ (\mathrm{LM}_\alpha) \text{ has no non-zero solution} \end{cases} \implies (1) \text{ has a solution.}$$

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For a given (M, g) with the positive Yamabe invariant and α large :

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Thank you for your attention !

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